On the logical structure of some choice, bar induction, maximality and well-foundedness principles equivalent to choice principles

Hugo Herbelin

based on joint work with Nuria Brede and Jad Koleilat

Verona

19 March 2024

Standard reverse mathematics of the axiom of choice in set theory

Three well-known equivalent presentations in set theory:

- **axiom of choice** (AC): any family of non-empty sets has a choice function
- **Zorn's lemma** (ZL): if all chains of a non-empty partially ordered set are bounded upwards, the set has a maximal belement
- **the well-ordering principle**: every set can be well-ordered

and many others:

- e.g. **Teichmüller-Tukey lemma**

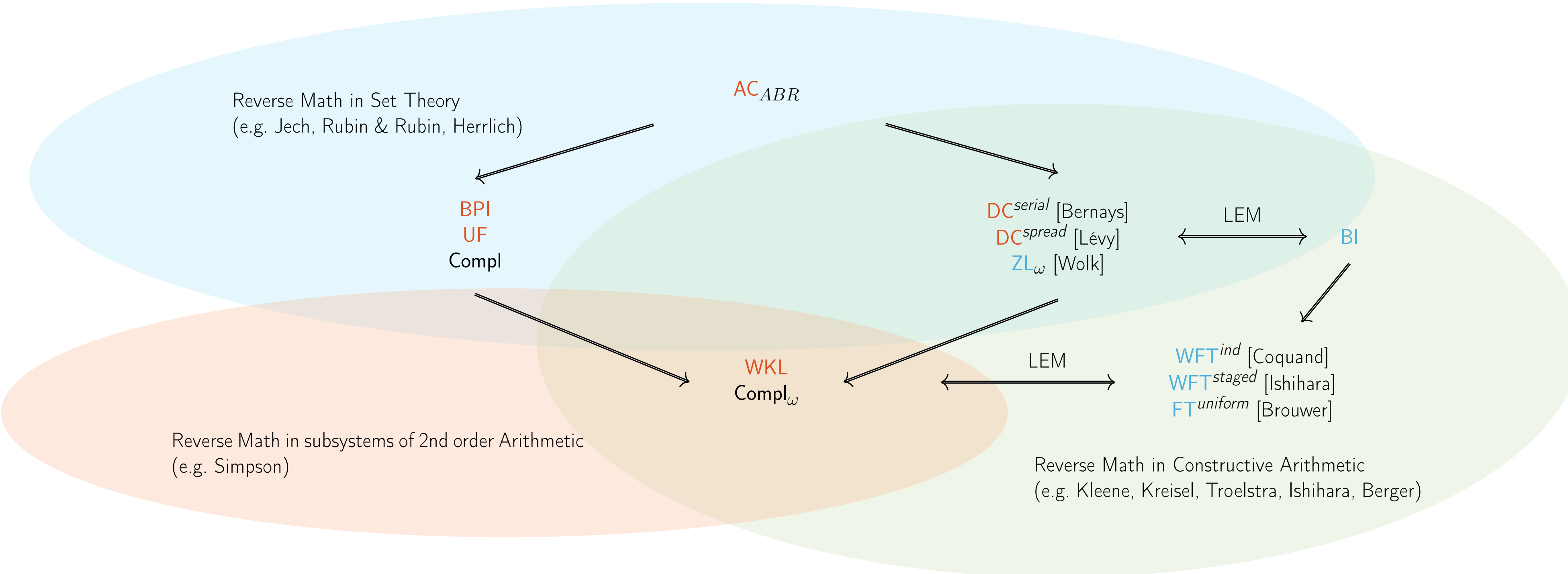
sometimes strictly weaker:

- axiom of **dependent** choice (DC), axiom of **countable** choice (AC_ω), Boolean prime ideal theorem (BPI), ultrafilter lemma (UF)

as well as variants in constructive mathematics, classically equivalent to choice or maximality principles:

- **bar induction**, its finite-branch version **fan theorem**, **update induction**, ...

Some standard results about the axiom of choice



BPI	=	Boolean Prime Ideal Theorem
UF	=	Ultrafilter Theorem
AC	=	Axiom of Choice
DC	=	Axiom of Dependent Choice
WKL	=	Weak Kőnig's Lemma

ZL_ω	=	Countable Zorn's Lemma
BI	=	Bar Induction
(W)FT	=	(Weak) Fan Theorem
Compl	=	Gödel's Completeness Theorem

Long-term objective: Look at the axiom of choice and its variants from a **logical** and **computational** perspective

The logical perspective:

- The axiom of choice and their variants assert the existence of **ideal** objects from intensional properties of these objects
- See e.g. Coquand's program of reformulating standard mathematical statements using equivalent **inductive** properties to avoid the axiom of choice

$\hookrightarrow$ some variants can indeed be seen as **extensionality** principles

$\hookrightarrow$ other variants as **well-foundedness** of processes producing arbitrarily precise approximations of ideal objects

Long-term objective: Look at the axiom of choice and its variants from a **logical** and **computational** perspective

The computational perspective:

- Following Brouwer, we know from Kolmogorov, Kleene, Curry, Howard, and many other that **intuitionistic proofs** are **programs**
- We know from Griffin 1990 that also **classical** proofs are **programs**, though they use "goto"-like side effects
- We know from works in Paris that proofs by **forcing** are **programs**, using a memory
- Other effects such as Lisp's **quote** are also useful to compute with some axioms (see Krivine, Pédrot, ...)
- More generally, it can be shown (by abstract reasoning) that any consistent mathematical axiom has an underlying computational content
- What is the **computational** content of the axiom of choice and its variants (Krivine's research programme)?

Contribution I

- A classification of choice and bar induction principles by means of two **dual** forms, seen as **extensionality principles**, for T a predicate filtering the finite approximations of functions from A to B:

Generalised Bar Induction (GBI_{ABT})

$$\underbrace{T\ A\text{-}B\text{-}\mathit{barred}}_{observational} \implies \underbrace{T\ A\text{-}B\text{-}\mathit{inductively\ barred}}_{effective}$$

Generalised Dependent Choice (GDC_{ABT})

$$\underbrace{T\ \mathit{coinductively}\ A\text{-}B\text{-}\mathit{approximable}}_{effective} \implies \underbrace{T\ \mathit{has\ an}\ A\text{-}B\text{-}\mathit{choice\ function}}_{observational}$$

- such that:

$\mathsf{GBI}_{\mathbb{N}BT}$	denotes	BI_{BT}
$\mathsf{GBI}_{\mathbb{N}\mathbb{B}\mathsf{ool}T}$	denotes	FT_{T}
$\mathsf{GDC}_{A\mathbb{B}\mathsf{ool}T}$	denotes	BPI_{AT}
$\mathsf{GDC}_{\mathbb{N}BT}$	has the strength of	DC_{BR}
$\mathsf{GDC}_{\mathbb{N}\mathbb{B}\mathsf{ool}T}$	has the strength of	WKL_{T}
GDC_{ABT}	has the strength of	AC_{ABR} for T "split"

Contribution II

- A pair of **dual** maximality and well-foundedness principles, for T a predicate filtering the finite approximations of functions from A to B:

Generalised Update Induction ($\mathbf{GUI}_{ABT}$)

(generalising Berger's update induction to arbitrary cardinals)

if the upwards monotone closure of T is $\prec$-inductive, it contains all functions from A to B

$\exists$ *Maximal Partial Choice Function* ($\exists\mathbf{MPCF}_{ABT}$)

(a functional variant of Teichmüller-Tukey's lemma)

if the downwards closure by restriction of T is non empty, it has a $\prec$-maximal partial choice function from A to B

where $\alpha \prec \beta$ is the approximation order on partial functions from A to B.

- such that: when A is $\mathbb{N}$, or B is $\mathbb{B}$**ool**, or T is split, coinductive approximability implies the totality of the choice function, recovering the previous statements, and dually for barredness.
- and such that: Zorn's Lemma, Teichmüller-Tukey's lemma, and other maximality principles are particular instances of $\exists\mathbf{MPCF}$.

Outline

Part A is organised around the following oppositions

- ill-founded (choice axioms) / well-founded (bar induction axioms)
- extensional (ideal object) / intensional (processus)
- closed by sequential restriction (= tree) / closed by sequential extension (= monotony)
- binary branching (B is $\mathbb{B}$ool) / finite branching (B is finite) / arbitrary branching (B is arbitrary)

Part B moves to arbitrary cardinals, so as to capture BPI and full AC

- sequential (A countable) / unordered (A arbitrary)
- closed by unordered restriction (= ideal) / closed by unordered extension (= filter)

Part C moves to maximality and well-foundedness principles

Part A

The sequential case: Kőnig's lemma, fan theorem, dependent choice, bar induction

What is bar induction?

Let's consider first different ways to define well-foundedness

Trees (and their negative) as predicates

Let B be a domain and u ranges over the set B^* of finite sequences of elements of B. We write $\langle\rangle$ for the empty sequence and $u \star b$ for the extension with one element. For T a **predicate** on B^*, we define:

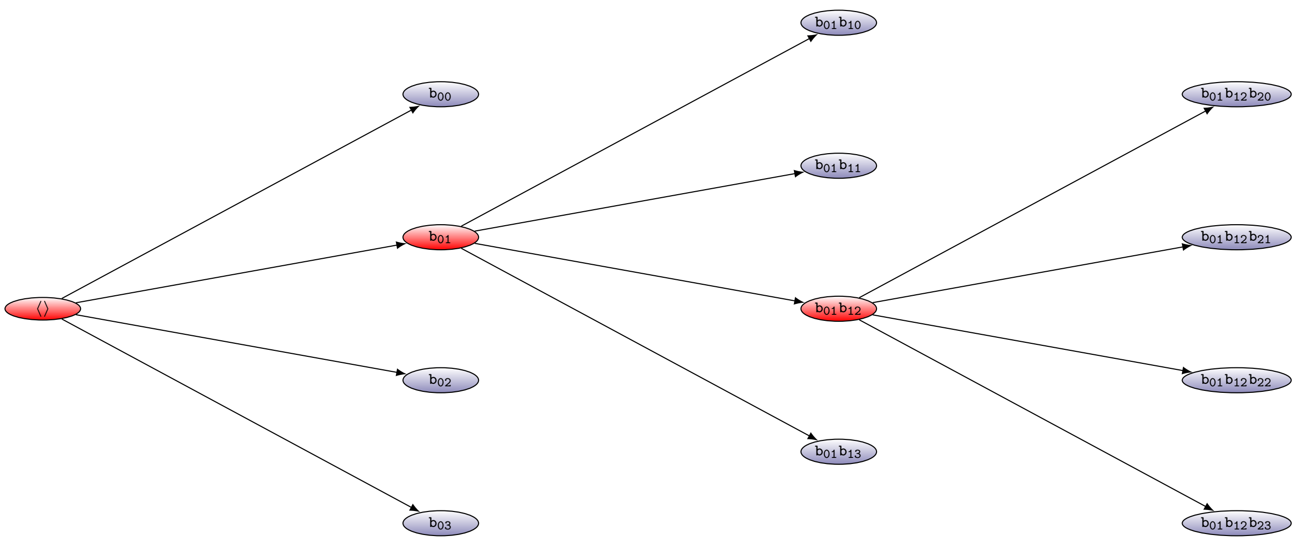
T is a tree	T is monotone
(closure under restriction)	(closure under extension)
$\forall u \, \forall a \, (u \star a \in T \Rightarrow u \in T)$	$\forall u \, \forall a \, (u \in T \Rightarrow u \star a \in T)$

Inductive characterisation of a well-founded tree-as-predicate

T *inductively well-founded* is short for inductively well-founded at $\langle\rangle \in A^*$

T *inductively well-founded at* u holds when:

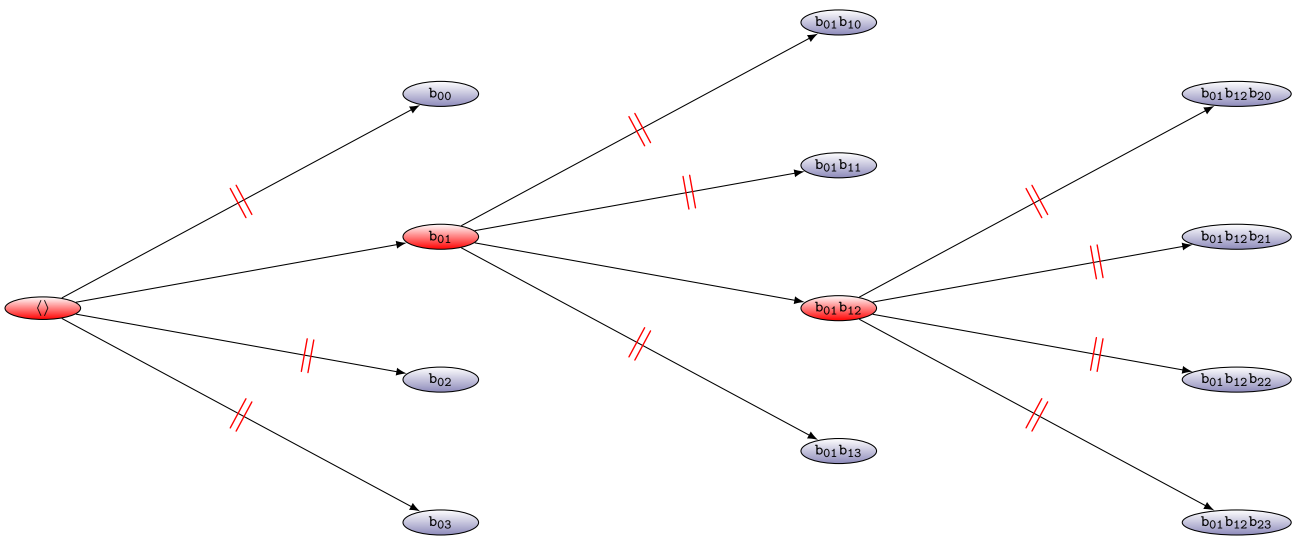
- $u \notin T$
- *or*, recursively, for all a, T is *inductively well-founded at* $u \star a$



Observational characterisation of a well-founded tree-as-predicate

T *observationally well-founded*

$$\forall \beta \in \mathbb{N} \to B.\ \exists n \in \mathbb{N}.\ \neg T(\beta_{|n})$$



Two characterisations of a well-founded tree-as-predicate

- From the **"effective"** representation of a well-founded tree we can always construct a predicate that is an **"observational"** representation of the tree
- To conversely obtain an effective representation of a tree T from its observational representation requires an axiom:

$$T \textit{ observationally well-founded} \implies T \textit{ inductively well-founded}$$

Bar Induction

If instead we build the **negative** of a tree-as-predicate and restate well-foundedness on the negative tree, one obtains bar induction:

- T *inductively well-founded* is the same as $\neg T$ *inductively barred*
- T *observationally well-founded* is the same $\neg T$ *barred*

- Bar Induction says that for a type B and a tree T,

$$\underbrace{T \text{ barred}}_{observational} \implies \underbrace{T \text{ inductively barred}}_{effective}$$

Dually: ill-foundedness

Dually, ill-foundedness of a tree T can be defined in different ways.

Let us concentrate on the finite-branching case. We have:

Effective view

$$T \text{ is staged infinite} \quad \triangleq \quad \forall n \, \exists u \, |u| = n \wedge u \in T$$

Observational view

$$T \text{ has an infinite branch} \quad \triangleq \quad \exists \alpha \, \forall u \leq \alpha \, T(u)$$

Kőnig's Lemma is a lemma that connects the two views when B is finite:

$$\mathsf{KL}_T \quad \triangleq \quad T \text{ is staged infinite} \Rightarrow T \text{ has an infinite branch}$$

Ill-foundedness, coinductively

Alternatively, by dualising the notion of inductively barred we get another **coinductive** definition of ill-foundedness, which we call productive. In full:

T productive is short for productive from $\langle\rangle \in B^*$

T productive from $u \in B^*$ holds when:

- u is in T
- *and*, recursively, there is $b \in B$ such that T is productive from $u \star b$

Relying on the notion of *inductively barred* and its dual, we obtain the following dual pair of choice and bar induction principles

Bar induction (BI_{BT})

T barred $\Rightarrow$ T inductively barred

Tree-Based Dependent Choice (DC_{BT}^{prod})

T productive $\Rightarrow$ T has an infinite branch

Recovering standard principles

$\mathsf{WKL}_T \iff \mathsf{DC}^{prod}_{\mathbb{B}\mathsf{ool}T}$ up to classical (actually co-intuitionistic) reasoning

$\mathsf{WFT}_T \iff \mathsf{BI}_{\mathbb{B}\mathsf{ool}T}$ up to intuitionistic reasoning

$\mathsf{DC}^{serial}_{BRb_0} \iff \mathsf{DC}^{prod}_{BR^{\triangleright}(b_0)}$

where

$$u \in R^{\triangleright}(b_0) \triangleq \text{case } u \text{ of } \begin{bmatrix} \langle\rangle & \mapsto & \top \\ b & \mapsto & R(b_0, b) \\ u' \star b \star b' & \mapsto & R(b, b') \end{bmatrix}$$

$$\mathsf{DC}^{serial}_{BRb_0} \triangleq \forall b \exists b' \, R(b, b') \Rightarrow \exists \alpha \, (\alpha(0) = b_0 \wedge \forall n \; R(\alpha(n), \alpha(n+1)))$$

(one of the most standard statement of dependent choice)

Part B

Relaxing the sequentiality

Relaxing the sequentiality

Let A and B be domains. Let now use v to range over the set $(A \times B)^*$ of finite sequences of pairs of elements in A and B.

We say $(a, b) \in v$ if (a, b) is one of the components of v.

We write $v \leq v'$ is v is included in v' when seen as sets.

For $v \in (A \times B)^*$, we write $dom(v)$ for the set of a such that there is some b such that $(a, b) \in v$.

If $\alpha \in A \rightarrow B$, we write $v \subset \alpha$ and say that v is a finite approximation of α if $\alpha(a) = b$ for all $(a, b) \in v$.

Let T be a predicate on $(A \times B)^*$. We write $\downarrow T$ and $\uparrow T$ to mean the following inner and outer closures with respect to $\leq$:

$$v \in \downarrow T \triangleq \forall v' \leq v \; (v' \in T)$$

$$v \in \uparrow T \triangleq \exists v' \leq v \; (v' \in T)$$

Relaxing the sequentiality (effective view)

T inductively A-B-barred from $v \in (A \times B)^*$ holds when:

- v is in the outer closure of T
- *or*, recursively, there exists $a \notin dom(v)$ such that for all $b \in B$, T is inductively A-B-barred from $v \star (a, b)$

T coinductively A-B-approximable from $v \in (A \times B)^*$ holds when:

- v is in the inner closure of T
- *and*, recursively, for all $a \notin dom(v)$, there is $b \in B$ such that T is coinductively A-B-approximable from $v \star (a, b)$

Relaxing the sequentiality (observational view)

T A-B-barred if $\forall\alpha \in A \to B \; \exists v \subset \alpha \; (v \in T)$

T has an A-B-choice function if $\exists\alpha \in A \to B \; \forall v \subset \alpha \; (v \in T)$

This leads to the following generalisation

Generalised Bar Induction (GBI_{ABT})

$$\underbrace{T\ A\text{-}B\text{-}\textit{barred}}_{observational} \implies \underbrace{T\ A\text{-}B\text{-}\textit{inductively barred}}_{effective}$$

Generalised Dependent Choice (GDC_{ABT})

$$\underbrace{T\ \textit{coinductively}\ A\text{-}B\text{-}\textit{approximable}}_{effective} \implies \underbrace{T\ \textit{has an}\ A\text{-}B\text{-}\textit{choice function}}_{observational}$$

Results justifying the generalisation

$$\mathsf{GBI}_{\mathbb{N}BT} \iff \mathsf{BI}_{BT}$$

$$\mathsf{GDC}_{\mathbb{N}BT} \iff \mathsf{DC}_{BT}^{prod}$$

The Boolean Prime Ideal Theorem

The specialisation to $\mathbb{B}$ool of the generalisation also captures the Boolean Prime Ideal Theorem.

Let $(\mathcal{B}, \vee, \wedge, \bot, \top, \neg, \vdash)$ be a Boolean algebra and I an ideal on $\mathcal{B}$. We extend I on $(\mathcal{B} \times \mathbb{B}\text{ool})^*$ by setting $u \in I^+$ if $(\bigvee_{(b,0)\in u} \neg b) \vee (\bigvee_{(b,1)\in u} b) \in I$. We have:

$$\mathsf{GDC}_{\mathcal{B}\mathbb{B}\text{ool}I^+} \iff \mathsf{BPI}_{\mathcal{B},I}$$

The full axiom of choice

Let AC$_{ABR}$ be $\forall a^A \, \exists b^B \, R(a,b) \Rightarrow \exists \alpha^{A \to B} \, \forall a^A \, R(a, \alpha(a))$

Define the *positive alignment* $R_\top$ of R by

$$R_\top \triangleq \lambda u.\, \forall (a,b) \in u \, R(a,b)$$

Then, AC$_{ABR}$ arrives as the instance GDC$_{ABR_\top}$

Strength of the generalisation

Without further restrictions, GDC and GBI are inconsistent:

- Take $A \triangleq \mathbb{N} \to \mathbb{B}\mathsf{ool}$
- Take $B \triangleq \mathbb{N}$
- Define T so that it constrains a choice function to be injective:

$$v \in T \triangleq \forall f\, f' n, ((f, n) \in v) \wedge ((f', n) \in v) \Rightarrow f = f'$$

Then, in the case of GDC, a coinductive A-B-approximation can always be found but an A-B-choice function would be an injective function from $\mathbb{N} \to \mathbb{B}\mathsf{ool}$ to $\mathbb{N}$, what is inconsistent.

A consistent restriction

A naive restriction is to require that:

- either A is countable
- or B is finite
- or T is **split** (or atomic, or unary), meaning for all u and v:
  - in the ill-founded case $u \in T \wedge v \in T \Rightarrow u \cup v \in T$
  - in the barred case $u \cup v \in T \Rightarrow u \in T \vee v \in T$

The restriction preserves the previous instantiations and makes GDC equivalent to AC since it implies AC, and, conversely, each of its three restrictions is implied by a consequence of AC.

Dually for GBI.

Summary of main results regarding choice and bar induction

$\mathsf{AC}_{ABR} \triangleq \mathsf{GDC}_{ABR_{\top}}$

$\mathsf{BPI}_{AI} \triangleq \mathsf{GDC}_{A\mathbb{B}\mathsf{ool}\mathcal{T}_I}$

$\mathsf{Compl}^{-}_{A\mathcal{T}} \triangleq \mathsf{GDC}_{A\mathbb{B}\mathsf{ool}\mathcal{T}^C}$

$\mathsf{Compl}^{+}_{A\mathcal{T}} \triangleq \mathsf{GBI}_{A\mathbb{B}\mathsf{ool}\mathcal{T}^C}$

$\mathsf{DC}^{\mathit{serial}}_{BRb} \triangleq \mathsf{GDC}_{\mathbb{N}BR^{\triangleright}_{\top}}$

$\mathsf{DC}^{\mathit{prod}}_{BT} \triangleq \mathsf{GDC}_{\mathbb{N}BT}$

$\mathsf{ZL}_{\mathbb{N}R} \triangleq \mathsf{GBI}_{\mathbb{N}B(\neg R)^{*}_{\perp}}$

$\mathsf{BI}_{BT} \triangleq \mathsf{GBI}_{\mathbb{N}BT}$

$\mathsf{WKL}^{\mathit{prod}}_{BT} \triangleq \mathsf{GDC}_{\mathbb{N}\mathbb{B}\mathsf{ool}T}$

$\mathsf{WFT}^{\mathit{ind}}_{BT} \triangleq \mathsf{GBI}_{\mathbb{N}\mathbb{B}\mathsf{ool}T}$

AC	=	Axiom of Choice
DC	=	Axiom of Dependent Choice
BPI	=	Boolean Prime Ideal Theorem
Compl^{-}	=	Completeness (consistent $\Rightarrow$ model)
WKL	=	Weak Kőnig's Lemma

Compl^{+}	=	Completeness (valid $\Rightarrow$ provable)
ZL	=	Zorn's Lemma
BI	=	Bar Induction
WFT	=	Weak Fan Theorem

Part C

Maximality and well-foundedness principles

A first solution to the inconsistency of the general form of GDC: requiring only a partial function

Generalised Partial Dependent Choice

$$\underbrace{T \text{ coinductively } A\text{-}B\text{-approximable}}_{\textit{effective}} \Longrightarrow \underbrace{T \text{ has a } \textbf{partial } A\text{-}B\text{-choice function}}_{\textit{observational}}$$

However, approximability happens to become a useless hypothesis, so such approach is not worth.

Teichmüller-Tukey Lemma

Let T be a predicate over A^*. We define its powerset closure by downwards restriction $\langle T\rangle$ as:

$$\langle T\rangle \quad \triangleq \quad \lambda\alpha^{\mathcal{P}(A)}.\forall u^{A^*}(u \subset \alpha \to u \in T)$$

Then, we say that a predicate P over predicates over A is of **finite character** if there is T such that $P = \langle T\rangle$.

Then, we can conversely rebuild T from $\langle T\rangle$ by setting

$$\begin{array}{lcl} \hat{u} & \triangleq & \lambda x^A.\ x \in u \\ \lfloor P\rfloor & \triangleq & \lambda u^{A^*}.\ \hat{u} \in P \end{array}$$

so that $T = \lfloor\langle T\rangle\rfloor$ and so that P is of finite character iff $P = \langle\lfloor P\rfloor\rangle$.

Teichmüller-Tukey Lemma

Teichmüller-Tukey is the statement that any non-empty predicate P of finite character has a maximal element with respect to inclusion.

$\exists$ Maximal Partial Choice Function ($\exists \mathbf{MPCF}_{ABT}$)

To make a connection with choice axioms, we introduce of variant of Teichmüller-Tukey lemma on functions: we consider now predicates over partial functions seen as predicates of finite character over the graph $\mathcal{G}(\alpha)$ of the function α, that is, as predicates over predicates over $A \times B$. Such predicates of finite character are generated by predicates T over $(A \times B)^*$. We can now state:

$\exists$ Maximal Partial Choice Function ($\exists \mathbf{MPCF}_{ABT}$)

if $\langle T \rangle$ is non empty, it has a $\prec$-maximal partial choice function from A to B

Or, fully formally:

$$(\exists \alpha (\mathcal{G}(\alpha) \in \langle T \rangle) \Rightarrow \exists \alpha^{A \rightharpoonup B} (\mathcal{G}(\alpha) \in \langle T \rangle \wedge \forall \beta \prec \alpha (\mathcal{G}(\beta) \notin \langle T \rangle))$$

where

$$\beta \prec \alpha \triangleq \begin{array}{c} \exists a^A (\alpha(a) = \bot \wedge \beta(a) \neq \bot) \\ \wedge\ \forall a (\alpha(a) \neq \bot \Rightarrow \beta(a) = \alpha(a)) \end{array}$$

$\exists\mathbf{MPCF}_{\mathbb{N}BT}$ is the contrapositive of Berger's update induction, and conversely, update induction can be generalised to arbitrary domains

P is of **finite character** over partial functions from $\mathbb{N}$ to B is the same as $\neg P$ **open predicate** in Berger's sense. This leads to the following:

Generalised Update Induction ($\mathbf{GUI}_{ABT}$)

if the upwards monotone closure of T is $\prec$-inductive, it contains all partial functions from A to B

Or, fully formally:

$$(\forall^{A\rightharpoonup B}\alpha(\forall\beta\prec\alpha(\mathcal{G}(\beta)\in\langle T\rangle^{\circ}))\Rightarrow(\mathcal{G}(\alpha)\in\langle T\rangle^{\circ}))\Rightarrow\forall\alpha(\mathcal{G}(\alpha)\in\langle T\rangle^{\circ})$$

where $\langle T\rangle^{\circ}$, an upwards monotone closure, is:

$$\langle T\rangle^{\circ} \;\triangleq\; \lambda\alpha^{\mathcal{P}(A\times B)}.\exists u^{(A\times B)^*}(u\subset\alpha\wedge u\in T)$$

We left implicit the definition of $A\rightharpoonup B$. It might typically be defined as $A\to B_{\perp}$ or as predicates over $A\times B$ that are functional. But both definitions are equivalent only up to classical reasoning...