Dualité du calcul et interprétations en terme de jeux

ANR Prelude

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Quelques repères et questions

Symétrie du calcul des séquents

$\hookrightarrow$ dualité termes/contextes d'évaluation

$\hookrightarrow$ la règle de coupure forme une paire critique

Modèles d'évaluation (élimination des coupures) basés sur la substitution

$\hookrightarrow$ deux manières asymétriques de résoudre la paire critique : appel par nom et appel par valeur

Peut-on mettre au point des modèles d'évaluation symétriques ?

$\hookrightarrow$ quels sont les calculs sous-jacents aux interprétations catégoriques symétriques de Došen, Lamarche et Straßburger, Straßburger, Pym et Führmann ?

Les stratégies des modèles en termes de jeux : des preuves sans coupure dans un calcul des séquents approprié

- coup d'Éloise (le joueur principal, le proposant) = règle d'introduction de connecteur positif
- coup d'Abélard (l'opposant) = règle d'introduction de connecteur négatif
- interaction entre joueurs = élimination de tête des coupures (asymétrique)
- peut-on voir toute preuve sans coupure de tout calcul des séquents comme une stratégie ?
- peut-on imaginer une interaction qui respecte la symétrie ?

L'approche du système $\mu\tilde{\mu}$

- two axioms

- no contraction : simulated by cuts with the axioms

- three kinds of sequents $\begin{cases} \text{terms : distinguished formula on the right} \\ \text{ev. contexts : distinguished formula on the left} \\ \text{commands : no distinguished formula} \end{cases}$

$$\frac{}{\Gamma, x : A \vdash x : A \,;\, \Delta}\, Ax_R \qquad \frac{}{\Gamma \,;\, \alpha : A \vdash \alpha : A, \Delta}\, Ax_L$$

$$\frac{c : (\Gamma \vdash \alpha : A, \Delta)}{\Gamma \vdash \mu\alpha.c : A \,;\, \Delta}\, \mu \qquad \frac{c : (\Gamma, x : A \vdash \Delta)}{\Gamma \,;\, \tilde{\mu}x.c : A \vdash \Delta}\, \tilde{\mu}$$

$$\frac{\Gamma \vdash v : A \,;\, \Delta \quad \Gamma \,;\, e : A \vdash \Delta}{\langle v \| e \rangle : (\Gamma \vdash \Delta)}\, Cut$$

Typing extensions (e.g. implication connective)

$$\frac{\Gamma, x : A \vdash v : B \,;\, \Delta}{\Gamma \vdash \lambda x.v : A \to B \,;\, \Delta} \qquad \frac{\Gamma \vdash v : A \,;\, \Delta \qquad \Gamma \,;\, e : B \vdash \Delta}{\Gamma \,;\, v \cdot e : A \to B \vdash \Delta}$$

Typing the $\mu\tilde{\mu}$-subsystem

(sequent calculus in context-free form)

Thanks to the absence of contraction, sequent calculus proofs can be represented *à la* natural deduction

$$
\dfrac{\begin{matrix}[A \vdash] \\ \vdots \\ \vdash\end{matrix}}{\vdash A}\,\mu
\qquad\qquad
\dfrac{\begin{matrix}[\vdash A] \\ \vdots \\ \vdash\end{matrix}}{A \vdash}\,\tilde{\mu}
$$

$$
\dfrac{\vdash A \qquad A \vdash}{\vdash}\,Cut
$$

$$
\dfrac{\begin{matrix}[\vdash A] \\ \vdots \\ \vdash B\end{matrix}}{\vdash A \to B}\,\to_R
\qquad\qquad
\dfrac{\vdash A \qquad B \vdash}{A \to B \vdash}\,\to_L
$$

The $\mu\tilde{\mu}$-subsystem

(the critical dilemma of computation)

Syntax

Commands	c	$::=$	$\langle v \Vert e \rangle$
Terms	v	$::=$	$\mu\alpha.c \mid x \mid \ldots$
Evaluation contexts	e	$::=$	$\tilde{\mu}x.c \mid \alpha \mid \ldots$

Semantics

$$
\begin{array}{lll}
(\mu) & \langle \mu\alpha.c \Vert e \rangle & \rightarrow c[\alpha \leftarrow e] \\
(\tilde{\mu}) & \langle v \Vert \tilde{\mu}x.c \rangle & \rightarrow c[x \leftarrow v]
\end{array}
$$

The critical pair

$$
\begin{array}{ccc}
 & \langle \mu\alpha.c \Vert \tilde{\mu}x.c' \rangle & \\
\swarrow (\mu) & & (\tilde{\mu}) \searrow \\
c[\alpha \leftarrow \tilde{\mu}x.c'] & & c'[x \leftarrow \mu\alpha.c]
\end{array}
$$

The $\mu\tilde{\mu}$-subsystem

(the critical dilemma of computation)

Syntax

Commands	c	$::=$	$\langle v \| e \rangle$
Terms	v	$::=$	$\mu\alpha.c \mid x \mid \ldots$
Evaluation contexts	e	$::=$	$\tilde{\mu}x.c \mid \alpha \mid \ldots$

Semantics

$$\begin{array}{lrcl} (\mu) & \langle \mu\alpha.c \| e \rangle & \rightarrow & c[\alpha \leftarrow e] \\ (\tilde{\mu}) & \langle v \| \tilde{\mu}x.c \rangle & \rightarrow & c[x \leftarrow v] \end{array}$$

The critical pair

call-by-value $\qquad \langle \mu\alpha.c \| \tilde{\mu}x.c' \rangle \qquad$ call-by-name

$$\swarrow (\mu) \qquad\qquad (\tilde{\mu}) \searrow$$

$$c[\alpha \leftarrow \tilde{\mu}x.c'] \qquad\qquad c'[x \leftarrow \mu\alpha.c]$$

The $\mu\tilde{\mu}$-subsystem

(the call-by-name confluent restriction)

Syntax

Commands	c	$::=$	$\langle v \| e \rangle$
Terms	v	$::=$	$\mu\alpha.c \mid x \mid \ldots$
Evaluation contexts	e	$::=$	$\tilde{\mu}x.c \mid E$
Linear ev. contexts	E	$::=$	$\alpha \mid \ldots$

Semantics

$$
\begin{array}{lrcl}
(\mu_n) & \langle \mu\alpha.c \| E \rangle & \rightarrow & c[\alpha \leftarrow E] \\
(\tilde{\mu}) & \langle v \| \tilde{\mu}x.c \rangle & \rightarrow & c[x \leftarrow v]
\end{array}
$$

The solved critical pair

$$
\begin{array}{ccc}
\langle \mu\alpha.c \| \tilde{\mu}x.c' \rangle & & \text{call-by-name} \\
& (\tilde{\mu}) \searrow & \\
& & c'[x \leftarrow \mu\alpha.c]
\end{array}
$$

The $\mu\tilde{\mu}$-subsystem

(the call-by-value confluent restriction)

Syntax

$$
\begin{array}{llll}
\text{Commands} & c & ::= & \langle v \| e \rangle \\
\text{Terms} & v & ::= & \mu\alpha.c \mid V \\
\text{Evaluation contexts} & e & ::= & \tilde{\mu}x.c \mid \alpha \mid \ldots \\
\text{Values} & V & ::= & x \mid \ldots
\end{array}
$$

Semantics

$$
\begin{array}{llll}
(\mu) & \langle \mu\alpha.c \| e \rangle & \rightarrow & c[\alpha \leftarrow e] \\
(\tilde{\mu}_v) & \langle V \| \tilde{\mu}x.c \rangle & \rightarrow & c[x \leftarrow V]
\end{array}
$$

The solved critical pair

call-by-value $\quad \langle \mu\alpha.c \| \tilde{\mu}x.c' \rangle$

$$\swarrow (\mu)$$

$$c[\alpha \leftarrow \tilde{\mu}x.c']$$

The $\mu\tilde{\mu}$-subsystem

(two confluent dual asymmetric restrictions)

$\mu_n\tilde{\mu}$-subsystem

Commands	c	$::=$	$\langle v \Vert e \rangle$
Terms	v	$::=$	$\mu\alpha.c \mid x \mid \ldots$
Linear ev. contexts	E	$::=$	$\alpha \mid \ldots$
Evaluation contexts	e	$::=$	$\tilde{\mu}x.c \mid E$

$$
\begin{array}{lccc}
(\mu_n) & \langle \mu\alpha.c \Vert E \rangle & \rightarrow & c[\alpha \leftarrow E] \\
(\tilde{\mu}) & \langle v \Vert \tilde{\mu}x.c \rangle & \rightarrow & c[x \leftarrow v]
\end{array}
$$

$\mu\tilde{\mu}_v$-subsystem

Commands	c	$::=$	$\langle v \Vert e \rangle$
Linear terms (= values)	V	$::=$	$x \mid \ldots$
Terms	v	$::=$	$\mu\alpha.c \mid V$
Evaluation contexts	e	$::=$	$\tilde{\mu}x.c \mid \alpha \mid \ldots$

$$
\begin{array}{lccc}
(\mu) & \langle \mu\alpha.c \Vert e \rangle & \rightarrow & c[\alpha \leftarrow e] \\
(\tilde{\mu}_v) & \langle V \Vert \tilde{\mu}x.c \rangle & \rightarrow & c[x \leftarrow V]
\end{array}
$$

Les E-dialogues de Lorenzen, d'après Felscher

(formes normales, logique classique)

Le calcul des E-dialogues de Lorenzen coïncide, dans le cas classique, avec LK^Q équipé avec les connecteurs négatifs $\rightarrow$, $\wedge_a$ et $\vee_m$.

Deux types de séquents : $\Gamma \vdash N; \Delta$ et $\Gamma \vdash; \Delta$

Interprétation possible en logique linéaire : $!\Gamma \vdash N; ?!\Delta$ et $!\Gamma \vdash; ?!\Delta$

$$\frac{\Gamma, A \vdash A; \Delta}{\Gamma, A \vdash; A, \Delta}\,Ax \qquad \frac{}{\Gamma, A \vdash A; \Delta}\,Ax \text{ si } A \text{ est atomique}$$

$$\frac{\Gamma, A \rightarrow B \vdash A; \Delta \quad \Gamma, A \rightarrow B, B \vdash; \Delta}{\Gamma, A \rightarrow B \vdash; \Delta} \rightarrow_g \qquad \frac{\Gamma, A \vdash; B}{\Gamma \vdash A \rightarrow B; \Delta} \rightarrow_d$$

$$\frac{\Gamma, A \wedge B, A \vdash; \Delta}{\Gamma, A \wedge B \vdash; \Delta} \wedge_g^1 \qquad \frac{\Gamma, A \wedge B, B \vdash; \Delta}{\Gamma, A \wedge B \vdash; \Delta} \wedge_g^2 \qquad \frac{\Gamma \vdash; A \quad \Gamma \vdash; B}{\Gamma \vdash A \wedge B; \Delta} \wedge_d$$

$$\frac{\Gamma, A \vee B, A \vdash; C, \Delta \quad \Gamma, A \vee B, B \vdash; C, \Delta}{\Gamma, A \vee B \vdash; C, \Delta} \vee_g \qquad \frac{\Gamma \vdash; A, B}{\Gamma \vdash A \vee B; \Delta} \vee_d$$

Les E-dialogues de Lorenzen, d'après Felscher

(formes normales annotées)

$$\frac{\Gamma, A \vdash V : A; \Delta}{\langle V \| \alpha \rangle : (\Gamma, A \vdash; \alpha : A, \Delta)}\ Ax \qquad \frac{}{\Gamma, x : A \vdash x : A; \Delta}\ Ax \text{ si } A \text{ est atomique}$$

$$\frac{\Gamma, z : A \to B \vdash V : A; \Delta) \quad c : (\Gamma, z : A \to B, y : B \vdash; \Delta)}{\langle z \| V \cdot \tilde{\mu} y.c \rangle : (\Gamma, z : A \to B \vdash; \Delta)} \to_g \qquad \frac{c : (\Gamma, x : A \vdash; \beta : B)}{\Gamma \vdash \lambda x.\mu\beta.c : A \to B; \Delta} \to_d$$

$$\frac{c : (\Gamma, z : A \wedge B, x : A \vdash; \Delta)}{\langle z \| \pi_1[\tilde{\mu} x.c] \rangle : (\Gamma, z : A \wedge B \vdash; \Delta)} \wedge_g^1 \quad \frac{c : (\Gamma, z : A \wedge B, y : B \vdash; \Delta)}{\langle z \| \pi_2[\tilde{\mu} y.c] \rangle : (\Gamma, z : A \wedge B \vdash; \Delta)} \wedge_g^2 \quad \frac{c : (\Gamma \vdash; \alpha : A) \quad c' : (\Gamma \vdash; \beta : B)}{\Gamma \vdash (\mu\alpha.c, \mu\beta.c') : A \wedge B; \Delta} \wedge_d$$

$$\frac{c : (\Gamma, z : A \vee B, x : A \vdash; C, \Delta) \quad c' : (\Gamma, z : A \vee B, y : B \vdash; C, \Delta)}{\langle z \| [\tilde{\mu} x.c, \tilde{\mu} y.c'] \rangle : \Gamma, z : A \vee B \vdash; C, \Delta} \vee_g \qquad \frac{c : (\Gamma \vdash; \alpha : A, \beta : B)}{\Gamma \vdash \mu(\alpha, \beta).c : A \vee B; \Delta} \vee_d$$

c	$::=$	$\langle x \| E\rangle \mid \langle V \| \alpha\rangle$	choix formule introduite	c	$::=$	$[^x_V \mid]^\alpha_E$
V	$::=$	$x \mid \lambda x.\mu\beta.c \mid (\mu\alpha_1.c, \mu\alpha_2.c) \mid \mu(\alpha_1, \alpha_2).c$	intro droite de négatif	V	$::=$	$A \to B \mid A \wedge B \mid A \vee B$
E	$::=$	$V \cdot \tilde{\mu}y.c \mid \pi_1[\tilde{\mu}x_1.c] \mid \pi_2[\tilde{\mu}x_2.c] \mid [\tilde{\mu}x_1.c, \tilde{\mu}x_2.c]$	intro gauche de négatif	E	$::=$	$V \mid \wedge_1 \mid \wedge_2 \mid \vee$

Les E-dialogues de Lorenzen, d'après Felscher

(interaction)

Règles de coupure

$$\frac{\Gamma \vdash A; \Delta \qquad \Gamma, A \vdash ; \Delta}{\Gamma \vdash ; \Delta}\ Coupe(hyp) \qquad \frac{\Gamma \vdash ; A, \Delta \qquad \Gamma, A \vdash ; \Delta}{\Gamma \vdash ; \Delta}\ Coupe(concl)$$

Syntaxe enrichie (coupure = substitutions explicites)

$$\begin{array}{lcl} c & ::= & c[\sigma] \\ [\sigma] & ::= & [\,] \parallel [x := V[\sigma]; \sigma] \parallel [\alpha := \tilde{\mu}x.c[\sigma]; \sigma] \end{array}$$

Règles d'élimination des coupures de tête (interaction)

$$\begin{array}{lcll} \langle x \| V \cdot \tilde{\mu}y.c \rangle [\sigma] & \longrightarrow & c'[x' := V[\sigma]; \alpha' := \tilde{\mu}y.c[\sigma]; \tau] & \sigma(x) \text{ est } \lambda x'.\mu\alpha'.c'[\tau] \\ \langle x \| \pi_1[\tilde{\mu}x_1.c_1] \rangle [\sigma] & \longrightarrow & c'_1[\alpha'_1 := \tilde{\mu}x_1.c_1[\sigma]; \tau] & \sigma(x) \text{ est } (\mu\alpha'_1.c'_1, \mu\alpha'_2.c'_2)[\tau] \\ \langle x \| \pi_2[\tilde{\mu}x_2.c_2] \rangle [\sigma] & \longrightarrow & c'_2[\alpha'_2 := \tilde{\mu}x_2.c_2[\sigma]; \tau] & \sigma(x) \text{ est } (\mu\alpha'_1.c'_1, \mu\alpha'_2.c'_2)[\tau] \\ \langle x \| [\tilde{\mu}x_1.c_1, \tilde{\mu}x_2.c_2] \rangle & \longrightarrow & c'[\alpha'_1 := \tilde{\mu}x_1.c_1[\sigma]; \alpha'_2 := \tilde{\mu}x_2.c_2[\sigma]; \tau] & \sigma(x) \text{ est } \mu(\alpha'_1, \alpha'_2).c'[\tau] \\ \langle V \| \alpha \rangle [\sigma] & \longrightarrow & c'[x' := V[\sigma]; \tau] & \sigma(\alpha) \text{ est } \tilde{\mu}x'.c'[\tau] \end{array}$$

+ règles pour le cas où V est x et le cas où $\sigma(x)$ ou $\sigma(\alpha)$ n'est pas défini.

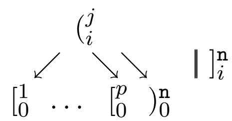
Remarque : il y a alternance entre le monde de c, x, α, σ et celui de c', x', α', τ.

Simply-typed $\mu_n\tilde{\mu}^{\to\mathbb{N}}$-system ($\mu$PCF)
(Herbelin [1997], Laird [1997])

$N \to N$ interpreted as $?N^{\perp}⅋N$
$\mathbb{N}$ interpreted as $?\oplus_n 1$
maximal $\eta_{\to n}$-expansion
maximal η_{μ}-expansion of atoms

$$c \;::=\; \langle x_i^j \| v_1 \cdot \ldots \cdot v_p \cdot [\mathtt{n} \mapsto c_{\mathtt{n}}]\rangle \mid \langle \mathtt{n} \| \alpha_i \rangle$$

where $v ::= \lambda x_1 \ldots x_n.\mu\alpha.c$

$c \;::=$ 

Initial state

$$\langle x \| \overbrace{v_1 \cdot \ldots \cdot v_p \cdot [\mathtt{n} \mapsto c_{\mathtt{n}}]}^{Opponent} \rangle \quad [x \leftarrow \overbrace{v}^{Player}]$$

Interaction rules

$$\begin{array}{llllll} (\to \mu_n) & \langle x_i^j \| \vec{v} \cdot [\mathtt{n} \mapsto c_{\mathtt{n}}]\rangle & [\sigma] & \to & c & [\vec{x} \leftarrow \vec{v}; \alpha \leftarrow [\mathtt{n} \mapsto c_{\mathtt{n}}]; \sigma'] \\ (\mathbb{N}) & \langle \mathtt{n} \| \alpha_i \rangle & [\sigma] & \to & c_{\mathtt{n}} & [\sigma'] \end{array}$$

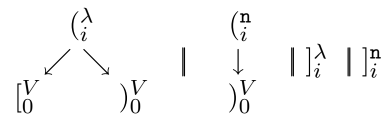
$$\sigma(x_i^j) = (\lambda\vec{x}.\mu\alpha.c)[\sigma'] \quad \sigma(\alpha_i) = ([\mathtt{n} \mapsto c_{\mathtt{n}}])[\sigma']$$

Simply-typed $\mu\tilde{\mu}_v^{\to\mathbb{N}}$-system ($\mu$PCF$_v$)
(Abramsky-McCusker [1997], Honda-Yoshida [1997], Laird [1998])

$P \to P$ interpreted as $P^{\perp}⅋!P$
$\mathbb{N}$ interpreted as $?\oplus_n 1$
maximal $\eta_{\to v}$-expansion and $\eta_{\tilde{\mu}}$-expansion
needs new constructions $⅄\mathtt{n}.v_{\mathtt{n}}$ and $\mathtt{n} \cdot e$

$$c \;::=\; \langle x_i \| V_\lambda \cdot e \rangle \mid \langle x_i \| \mathtt{n} \cdot e \rangle \mid \langle V_\lambda \| \alpha_i \rangle \mid \langle \mathtt{n} \| \alpha_i \rangle$$

where $V_\lambda ::= \lambda x.\mu\alpha.c \mid ⅄\mathtt{n}.\mu\alpha.c_{\mathtt{n}}$
$e ::= \tilde{\mu}x.c \mid [\mathtt{n} \mapsto c_{\mathtt{n}}]$

$c \;::=$ 

Initial states

$$\begin{array}{ll} \langle \overbrace{\mathtt{n}}^{Player} \| \alpha \rangle & [\alpha \leftarrow \overbrace{[\mathtt{n} \mapsto c_{\mathtt{n}}]}^{Opponent}] \\ \langle \lambda x.\mu\alpha.c \| \alpha \rangle & [\alpha \leftarrow V_\lambda \cdot e] \\ \langle ⅄\mathtt{n}.\mu\alpha.c_{\mathtt{n}} \| \alpha \rangle & [\alpha \leftarrow \mathtt{n} \cdot e] \end{array}$$

Interaction rules

$$\begin{array}{llllll} (\to \mu) & \langle x_i \| V_\lambda \cdot e \rangle & [\sigma] & \to & c & [x \leftarrow V_\lambda; \alpha \leftarrow e; \sigma'] \\ (\to^{\mathbb{N}} \mu) & \langle x_i' \| \mathtt{n} \cdot e \rangle & [\sigma] & \to & c_{\mathtt{n}} & [\alpha \leftarrow e; \sigma'] \\ (\tilde{\mu}_v) & \langle V_\lambda \| \alpha_i \rangle & [\sigma] & \to & c & [x \leftarrow V_\lambda; \sigma'] \\ (\mathbb{N}) & \langle \mathtt{n} \| \alpha_i' \rangle & [\sigma] & \to & c_{\mathtt{n}} & [\sigma'] \end{array}$$

$$\sigma(x_i) = (\lambda x.\mu\alpha.c)[\sigma'] \qquad \sigma(\alpha_i) = (\tilde{\mu}x.c)[\sigma']$$
$$\sigma(x_i') = (⅄\mathtt{n}.\mu\alpha.c_{\mathtt{n}})[\sigma'] \qquad \sigma(\alpha_i') = ([\mathtt{n} \mapsto c_{\mathtt{n}}])[\sigma']$$

A general, purely computational, definition of connective

A connective is the pair of a family of finite sequences of signs (standing for a family of constructors and the signs telling if the arguments are terms or evaluation contexts) and of a sign (telling if the connective "constructs" a term or an evaluation term). Here are examples :

conjunctive connectives :	$\wedge_m$	$\{++\}+$	$\wedge_a$	$\{\dot{-},\dot{-}\}-$				
subtractive connectives :	$\diagdown$	$\{+\dot{-}\}+$	$\diagdown'$	$\{+-\}+$	$\diagup$	$\{\dot{-}+\}+$	$\diagup'$	$\{-+\}+$
disjunctive connectives :	$\vee_m$	$\{--\}-$	$\vee_a$	$\{\dot{+},\dot{+}\}+$	$\vee_m^R$	$\{-\dot{-}\}-$	$\vee_m^L$	$\{\dot{-}-\}-$
implication connectives :	$\rightarrow$	$\{+\dot{-}\}-$	$\rightarrow'$	$\{+-\}-$	$\leftarrow$	$\{\dot{-}+\}-$	$\leftarrow'$	$\{-+\}-$
true connective :	$\top_a$	$\{\}+$	$\top_m$	$\{\epsilon\}-$	ϵ denotes the empty sequence			
false connective :	$\bot_m$	$\{\epsilon\}-$	$\bot_a$	$\{\}-$	ϵ denotes the empty sequence			
identity :	$\neg$	$\{\dot{+}\}+$			whose constructors are isomorphic to those of $\{\dot{-}\}-$			
negation :	$\neg$	$\{\dot{-}\}+$			whose constructors are isomorphic to those of $\{\dot{+}\}-$			
negating conjunctions :	$\overline{\vee}_m$	$\{++\}-$	$\overline{\vee}_a$	$\{\dot{-},\dot{-}\}+$				
negated disjunctions :	$\overline{\wedge}_m$	$\{--\}+$	$\overline{\wedge}_a$	$\{\dot{+},\dot{+}\}-$				
quantifiers :	$\exists$	$\{\dot{+}\}_i+$	$\forall$	$\{\dot{-}\}_i-$	where i ranges over some domain of terms			
negating quantifiers :	$\exists\neg$	$\{\dot{-}\}_i+$	$\forall\neg$	$\{\dot{+}\}_i-$	where i ranges over some domain of terms			
ludic's pos. connective :	$\oplus_{I\subset_{\text{fin}}\mathbb{N}}\otimes_I$	$\{\overbrace{+\ldots+}^{n\text{ times}}\}_{\{i_1,\ldots,i_n\}}+$			where $\{i_1,\ldots,i_n\}$ ranges over finite subsets of $\mathbb{N}$			
ludic's neg. connective :	$\&_{I\subset_{\text{fin}}\mathbb{N}}⅋_I$	$\{\overbrace{-\ldots-}^{n\text{ times}}\}_{\{i_1,\ldots,i_n\}}-$			where $\{i_1,\ldots,i_n\}$ ranges over finite subsets of $\mathbb{N}$			
ludic's modified p. conn. :	$\oplus_{n,j\in\mathbb{N}}\otimes_{[1;n]}$	$\{\overbrace{+\ldots+}^{n\text{ times}}\}_{n,j}+$			which differs only in that the component $\otimes_\emptyset$ in			
ludic's modified n. conn. :	$\&_{n,j\in\mathbb{N}}⅋_{[1;n]}$	$\{\overbrace{-\ldots-}^{n\text{ times}}\}_{n,j}-$			$\oplus_{I\subset_{\text{fin}}\mathbb{N}}\otimes_I$ has now arbitrary many instances			

Ludics syntax

(a slight generalization, syntax)

Cut-free syntax

Commands	c	$::=$	$\langle x \| n\ p\ V_1 \dots V_n \rangle \ \mid\ \Omega\ \mid\ \dagger$
Terms	V	$::=$	$\boldsymbol{\lambda} n.\boldsymbol{\lambda} p.\lambda x_1 \dots x_n.c$

"Dessein" $=$ x's occur linearly and p branching is degenerated when $n = 0$ (since there is only one empty subset while there are infinitely many finite subset of non-zero cardinal – p is the index of a subset in an enumeration of finite subsets of cardinal n)

Remark : To avoid variable captures, names are made of sequences of integers. In $\langle x \| n\ p\ V_1 \dots V_n \rangle$, if V_i is $\boldsymbol{\lambda} n.\boldsymbol{\lambda} p.\lambda x_1 \dots x_n.c_p^n$ then x_j is the concatenation of j to the name x.

Remark : The generalization allows for instance to represent *true* $:= \langle x \| 0\ 0 \rangle : 1 \oplus 1$ and *false* $:= \langle x \| 0\ 1 \rangle : 1 \oplus 1$ (where x is the name of the formula $1 \oplus 1$).

Remark : one could also have used the abbreviation $E ::= n\ p\ V_1 \dots V_n$ and $c ::= \langle x \| E \rangle$ to make a closer relation with the syntax used for E-dialogues.

Ludics syntax

(a slight generalization, semantics)

Syntax with cuts

Commands	c	$::=$	$\langle x \| n\ p\ V_1 \ldots V_n \rangle \mid \Omega \mid \dagger \mid c[\sigma]$
Terms	V	$::=$	$\boldsymbol{\lambda} n.\boldsymbol{\lambda} p.\lambda x_1 \ldots x_n.c$
Substitutions	$[\sigma]$	$::=$	$[\,] \mid [x := V[\sigma]; \sigma]$

Semantics

Weak-head cut-elimination :

$\langle x \| n\ p\ V_1 \ldots V_n \rangle[\sigma]$	$\rightarrow$	$c_p^n[x_1 := V_1[\sigma]; \ldots ; x_n := V_n[\sigma]; \tau]$	if $\sigma(x)$ is $\boldsymbol{\lambda} n.\boldsymbol{\lambda} p.\lambda x_1 \ldots x_n.c_p^n[\tau]$
$\Omega[\sigma]$	$\rightarrow$	Ω	failure
$\dagger[\sigma]$	$\rightarrow$	$\dagger$	final step

Strong reduction (weak-head cut-elimination below constructors) :

$\langle x \| n\ p\ V_1 \ldots V_n \rangle[\sigma] \ \rightarrow\ \langle x \| n\ p\ V'_1 \ldots V'_n \rangle$ if x not bound in σ

where V_i is $\boldsymbol{\lambda} n.\boldsymbol{\lambda} p.\lambda x_1 \ldots x_n.c_{np}$ and $c_{np}[\sigma] \rightarrow c'_{np}$ and V'_i is $\boldsymbol{\lambda} n.\boldsymbol{\lambda} p.\lambda x_1 \ldots x_n.c'_{np}$

Ludics as a sequent calculus

(a slight generalization, typing)

Types

$$N \quad ::= \quad \&_{I \in \mathcal{I}} \wp_{i \in I} N_{Ii}^{\perp} \qquad \text{where } \mathcal{I} \subset \mathcal{P}_{fin}(\mathbb{N})$$

Typing

$$\frac{\Delta \vdash V_1 : N_{I_0 i_1} \quad \ldots \quad \Delta \vdash V_n : N_{I_0 i_n}}{\langle x \| n\, (\#_n I_0)\, V_1 \, \ldots \, V_n \rangle : (\Delta, x : \&_{I \in \mathcal{I} \vdash} \wp_{i \in I} N_{Ii}^{\perp})} \; I_0 = \{i_1 \ldots i_n\}$$

$$\frac{\forall I = \{i_1 \ldots i_n\} \in \mathcal{I} \quad c^n_{\#_n I} : (\Delta, x_1 : N_{Ii_1}, \ldots, x_n : N_{Ii_n} \vdash)}{\Delta \vdash \boldsymbol{\lambda} n . \boldsymbol{\lambda} p . \lambda x_1 \ldots x_n . c^n_p : \&_{I \in \mathcal{I}} \wp_{i \in I} N_{Ii}^{\perp}} \; \mathcal{I}$$

where $c^n_{\#_n I}$ is Ω if $I \notin \mathcal{I}$

$$\frac{}{\dagger : (\vdash \Delta)}$$

$$\frac{\ldots \; \Delta \vdash V_i : N_i \; \ldots \qquad c : (\Delta, x_1 : N_1, \ldots, x_n : N_n \vdash)}{c[x_1 := V_1; \ldots ; x_n := V_n] : (\Delta \vdash)}$$

$$\frac{\ldots \; \Delta \vdash V_i : N_i \; \ldots \qquad \Delta, x_1 : N_1, \ldots, x_n : N_n \vdash V : N}{\Delta \vdash V[x_1 := V_1; \ldots ; x_n := V_n] : N}$$

($\#_n I$ is the index of $I \in \mathbb{N}^n$ in an enumeration of $\mathbb{N}^n$)

Ludics as a $\mu\tilde{\mu}^{(\&\wp)}$-calculus

Syntax

Commands	c	$::=$	$\langle V \| e\rangle \mid \Omega \mid \dagger$
Terms	V	$::=$	$x \mid \boldsymbol{\lambda} n.\boldsymbol{\lambda} p.\lambda x_1 \ldots x_n.c$
Evaluation contexts	e	$::=$	$n\ p\ V_1 \ldots V_n \mid \tilde{\mu}x.c$

Semantics

$$\langle \boldsymbol{\lambda} n.\boldsymbol{\lambda} p.\lambda x_1 \ldots x_n.c_p^n \| n\ p\ V_1 \ldots V_n\rangle \rightarrow c_p^n[V_1/x_1 \ldots V_n/x_n]$$
$$\langle V \| \tilde{\mu}x.c\rangle \rightarrow c[V/x]$$

Typing

$$\frac{\Delta \vdash V_1 : N_{I_0 i_1} \quad \ldots \quad \Delta \vdash V_n : N_{I_0 i_n}}{\Delta; n\,(\#_n I_0)\, V_1 \ldots V_n : \&_{I \in \mathcal{I} \vdash} \wp_{i \in I} N_{Ii}^{\perp}}\ I_0 = \{i_1 \ldots i_n\}$$

$$\frac{\forall I = \{i_1, \ldots, i_n\} \in \mathcal{I} \quad c^n_{\#_n I} : (\Delta, x_1 : N_{Ii_1}, \ldots, x_n : N_{Ii_n} \vdash)}{\Delta \vdash \boldsymbol{\lambda} n.\boldsymbol{\lambda} p.\lambda x_1 \ldots x_n.c_p^n : \&_{I \in \mathcal{I}} \wp_{i \in I} N_{Ii}^{\perp}}\ \mathcal{I}$$

where $c^n_{\#_n I}$ is Ω if $I \notin \mathcal{I}$

$$\frac{c : (\Delta, x : N \vdash)}{\Delta; \tilde{\mu}x.c : N \vdash} \qquad \frac{\Delta \vdash V : N \qquad \Delta; e : N \vdash}{\langle V \| e\rangle : (\Delta \vdash)} \qquad \frac{}{\Omega : (\vdash \Delta)} \qquad \frac{}{\dagger : (\vdash \Delta)} \qquad \frac{x : N \text{ in } \Delta}{\Delta \vdash x : N}$$

($\#_n I$ is the index of $I \in \mathbb{N}^n$ in an enumeration of $\mathbb{N}^n$)