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AN ANALYSIS OF THE CONSTRUCTIVE CONTENT OF HENKIN'S PROOF OF GÖDEL'S COMPLETENESS THEOREM

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Abstract. Gödel's completeness theorem for classical first-order logic is one of the most basic theorems of logic. Central to any foundational course in logic, it connects the notion of valid formula to the notion of provable formula.

We survey a few standard formulations and proofs of the completeness theorem before focusing on the formal description of a slight modification of Henkin's proof within intuitionistic second-order arithmetic.

It is standard in the context of the completeness of intuitionistic logic with respect to various semantics such as Kripke or Beth semantics to follow the Curry-Howard correspondence and to interpret the proofs of completeness as programs which turn proofs of validity for these semantics into proofs of derivability.

We apply this approach to Henkin's proof to phrase it as a program which transforms any proof of validity with respect to Tarski semantics into a proof of derivability.

By doing so, we hope to shed an "effective" light on the relation between Tarski semantics and syntax: proofs of validity are syntactic objects that we can manipulate and compute with, like ordinary syntax.

CONTENTS

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§1. Preliminaries.

1.1. The completeness theorem. The completeness theorem for classical first-order logic is one of the most basic and traditional theorems of logic. Proved by Gödel in 1929 [34] as an answer to a question raised by Hilbert and Ackermann in 1928 [45], it states that any of the standard equivalent formal systems for defining provability in first-order logic is complete enough to include a derivation of every valid formula. A formula A is *valid* when it is true under all interpretations of its primitive symbols over any domain of quantification.

Let $\mathcal{L}$ be a signature for first-order logic, i.e. the data of a set[1] $\mathcal{Fun}$ of function symbols, each of them coming with an arity, as well as of a set $\mathcal{Pred}$ of predicate symbols, each of them also coming with an arity. We call constants the function symbols of arity 0 and propositional atoms the predicate symbols of arity 0. When studying the computational content of Gödel's completeness in Section 2, we shall restrict the language to a countable one but the rest of this section does not require restrictions on the cardinal of the language.

We let f range over $\mathcal{Fun}$ and P range over $\mathcal{Pred}$. For $f \in \mathcal{Fun}$ and $P \in \mathcal{Pred}$, we respectively write their arity a_f and a_P. Let x range over a countable set $\mathcal{X}$ of variables and let t range over the set $\mathcal{Term}$ of terms over $\mathcal{L}$ as described by the following grammar:

$$t \quad ::= \quad x \mid f(t_1, ..., t_{a_f})$$

Let A range over the set $\mathcal{Form}$ of formulae over $\mathcal{L}$ as described by the following grammar:

$$A \quad ::= \quad P(t_1, ..., t_{P_f}) \mid \dot{\bot} \mid A \mathrel{\dot{\Rightarrow}} A \mid \dot{\forall} x\, A$$

Note that in classical first-order logic, the language of *negative* connectives and quantifiers made of $\dot{\Rightarrow}$, $\dot{\forall}$ and $\dot{\bot}$ is enough to express all other connectives and quantifiers. The dot over the notations is to distinguish the connectives and quantifier of the logic we are talking about (object logic) from the connectives and quantifiers of the ambient logic in which the completeness theorem is formulated (meta-logic, see below). We take $\dot{\bot}$ as a primitive connective and this allows to express consistency of the object logic as the non-provability of $\dot{\bot}$. Negation can then be defined as $\dot{\neg} A \triangleq A \mathrel{\dot{\Rightarrow}} \dot{\bot}$. Also, in $\dot{\forall} x\, A$, we say that x is a binding variable which binds all occurrences of x in A (if any). If the occurrence of a variable is not in the scope of a $\dot{\forall}$ with same name, it is called free. If a formula has no free variables, we say it is *closed*.

Let us write Γ for finite contexts of hypotheses, as defined by the following grammar:

$$\Gamma \quad ::= \quad \epsilon \mid \Gamma, A$$

In particular, ϵ denotes the empty context, which we might also not write at all, as e.g. in $\vdash A$ standing for $\epsilon \vdash A$.

We assume having chosen a formal system for provability in classical first-order logic, e.g. one of the axiomatic systems given in Frege [29] or in Hilbert and Ackermann [45], or one of the systems such as Gentzen-Jaśkowski's natural deduction [53, 33] or Gentzen's sequent calculus [33], etc., and we write $\Gamma \vdash A$ for the statement that A is provable under the finite context of hypotheses Γ. If $\mathcal{M}$ is a model for classical logic and σ an interpretation of the variables from $\mathcal{X}$ in the model, we write $\mathcal{M} \models_\sigma A$ for the statement expressing that A is true in the model $\mathcal{M}$ (to be defined

[1] We use here "set" in an informal way, not necessarily assuming the metalanguage to be specifically set theory.

in Section 1.4). *Validity* of A under assumptions Γ, written $\Gamma \vDash A$ is defined to be $\forall \mathcal{M}\ \forall \sigma\ (\mathcal{M} \vDash_\sigma \Gamma \Rightarrow \mathcal{M} \vDash_\sigma A)$ where $\mathcal{M} \vDash_\sigma \Gamma$ is the conjunction of all $\mathcal{M} \vDash_\sigma B$ for every B in Γ, i.e. $\bigwedge_{B \in \Gamma} \mathcal{M} \vDash_\sigma B$. Note that $\Rightarrow$, $\forall$, $\wedge$ and, later on, below, $\vee$, $\exists$, $\perp$, as well as derived $\neg$, represent the connectives and quantifiers of the metalanguage.

We say that Γ *is inconsistent* if $\Gamma \vdash \perp$ and *consistent* if $\Gamma \nvdash \perp$, i.e. if $(\Gamma \vdash \perp) \Rightarrow \perp$, i.e., if a contradiction in the object language is reflected as a contradiction in the metalogic. We say that Γ *has a model* if there exist $\mathcal{M}$ and σ such that $\mathcal{M} \vDash_\sigma \Gamma$. The completeness theorem, actually a weak form of the completeness theorem as discussed in the next section, is commonly stated under one of the following classically but not intuitionistically equivalent forms:

$$\begin{array}{ll} C1. & \vDash A \ \Rightarrow \vdash A \\ C2. & \Gamma \text{ is consistent } \Rightarrow \ \Gamma \text{ has a model} \\ C3. & \Gamma, \neg A \text{ has a model } \vee \ \Gamma \vdash A \end{array}$$

1.2. Weak and strong completeness. In a strong form, referred to as strong completeness[2], completeness states that any formula valid under some possibly infinite theory is provable under a finite subset of this theory. This is the most standard formulation of completeness in textbooks, and, as such, it is a key component of the compactness theorem. Also proved by Gödel [35], the compactness theorem states that it is enough for a theory to have a model that any finite subset of the theory has a model. In contrast, completeness with respect to finite theories as stated above is referred to as weak completeness. Let $\mathcal{T}$ be a set of formulae and let $\mathcal{T} \vdash A$ mean the existence of a finite sequence Γ of formulae in $\mathcal{T}$ such that $\Gamma \vdash A$. Let $\mathcal{M} \vDash_\sigma \mathcal{T}$ be $\forall B \in \mathcal{T}\ \mathcal{M} \vDash_\sigma B$ and let the definitions of $\mathcal{T}$ is consistent and of $\mathcal{T}$ has model be extended accordingly. The strong formulations of the three views at weak completeness above are now the following:

$$\begin{array}{ll} S1. & \mathcal{T} \vDash A \ \Rightarrow \ \mathcal{T} \vdash A \\ S2. & \mathcal{T} \text{ is consistent } \Rightarrow \ \mathcal{T} \text{ has a model} \\ S3. & \mathcal{T} \cup \{\neg A\} \text{ has a model } \vee \ \mathcal{T} \vdash A \end{array}$$

We shall consider the formalisation and computational content of strong completeness. Weak completeness will then come as a special case.

1.3. The standard existing proofs of completeness. Let us list a few traditional proofs from the classic literature[3].

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[2]We follow here a terminology dubbed by Henkin in his 1947 dissertation, according to [40]. However, in the context of intuitionistic logic, some authors use the weak and strong adjectives with different meanings. For instance, in Kreisel [59, 60], the statement $(\vDash A) \Rightarrow (\vdash A)$ is called strong completeness while weak completeness is the statement $(\vDash A) \Rightarrow \neg\neg(\vdash A)$. In the context of semantic cut-elimination, e.g. in Okada [70], $(\vDash A) \Rightarrow (\vdash A)$ is only a weak form of completeness whose strong form is the statement $(\vDash A) \Rightarrow (\vdash_{\text{cut-free}} A)$, for a notion of cut-free proof similar to the notion of cut-free proof in Gentzen's sequent calculus or to normal proofs in Prawitz' analysis of normalisation for natural deduction.

[3]We cite the most common proofs in the classic pre-1960 literature. Recent developments include e.g. Joyal's categorical presentation of a completeness theorem. We can also cite Berger's [74, Sec. 1.4.3] or Krivtsov [66] construction in intuitionistic logic of a classical model from a Beth model for classical provability. These two latter proofs are variants of the Beth-Hintikka-Kanger-Schütte style of proofs, the first one relying on the axiom of dependent choice and the second on the (weaker) Fan theorem.

- Gödel's original proof [34] considers formulae in prenex form and works by induction on the number of quantifiers for reducing the completeness of first-order predicate logic completeness to the completeness of propositional logic.
- Henkin's proof [39] is related to statement S2: from the assumption that $\mathcal{T}$ is consistent, a syntactic model over the terms is built as a maximal consistent extension of $\mathcal{T}$ obtained by ordering the set of formulae and extending $\mathcal{T}$ with those formulae that preserve consistency, following the ordering.
- In the 1950's, a new kind of proof independently credited to Beth [11], Hintikka [47, 48], Kanger [56] and Schütte [75] was given. The underlying idea is to build an infinite normal derivation, typically in sequent calculus. Rules are applied in a fair way, such that all possible combinations of rules are considered. If the derivation happens to be finite, a proof is obtained. Otherwise, by weak Kőnig's lemma, there is an infinite branch and this infinite branch gives rise to a countermodel. The intuition underlying this proof is then best represented by statement S3.
- In the 1950's also, Rasiowa and Sikorski [72] gave a variant of Henkin's proof relying on the existence of an ultrafilter for the Lindenbaum algebra of classes of logically equivalent formulae, identifying validity with having value 1 in all interpretations of a formula within the two-value Boolean algebra $\{0, 1\}$. This is close to Henkin's proof in the sense that Henkin's proof implicitly builds an ultrafilter of the Lindenbaum algebra of formulae.

Our main contribution in this paper is the analysis in Section 2 of the computational content of Henkin's proof.

1.4. Models and truth. The interpretation of terms in a model $\mathcal{M}$ is given by a domain $\mathcal{D}$ and by an interpretation $\mathcal{F}$ of the symbols in $\mathcal{F}un$ such that $\mathcal{F}(f) \in \mathcal{D}^{a_f} \to \mathcal{D}$, where $\mathcal{D}^{a_f} \to \mathcal{D}$ denotes the set of functions of arity a_f over $\mathcal{D}$. Then, given an assignment $\sigma \in \mathcal{X} \to \mathcal{D}$ of the variables to arbitrary values of the domain, the interpretation of terms in $\mathcal{D}$ is given by:

$$\begin{array}{lcl} [\![x]\!]^\sigma_{\mathcal{M}} & \triangleq & \sigma(x) \\ [\![f(t_1, \ldots, t_{a_f})]\!]^\sigma_{\mathcal{M}} & \triangleq & \mathcal{F}(f)([\![t_1]\!]^\sigma_{\mathcal{M}}, \ldots, [\![t_{a_f}]\!]^\sigma_{\mathcal{M}}) \end{array}$$

To interpret formulae, two common approaches are used in the literature.

- *Tarski semantics (predicates as predicates).* This is the approach followed e.g. in the Handbook of Mathematical Logic [6], the Handbook of Proof Theory [14] or in the original proof of Gödel [34]. This approach interprets formulae of the object language propositionally, i.e. as formulae of the metalanguage. In this case, the interpretation depends on whether the metalanguage is classical or not. For instance, in a classical metalanguage, the theory

$$Classic \triangleq \{\dot\neg\dot\neg A \mathbin{\dot\Rightarrow} A \mid A \in \mathcal{F}orm\}$$

  would be true in all models. On the other hand, in an intuitionistic metalanguage, a formula such as, say, $\dot\neg\dot\neg X \mathbin{\dot\Rightarrow} X$ could not be proved true in all models[4]. In a

[4]For instance, if coh_{ML} is the formula expressing the consistency of the metalanguage represented as an object language in the metalanguage itself, then a model $\mathcal{M}$ binding atom X to the metalanguage formula $coh_{ML} \vee \neg coh_{ML}$ would intuitionistically satisfy $\mathcal{M} \models \neg\neg X$ but not $\mathcal{M} \models X$.

strongly anti-classical intuitionistic metalanguage refuting double-negation elimination, it could even be proved that there are models[5] which refute $\dot\neg\dot\neg X \dot\Rightarrow X$.

The possible presence of models provably anti-classical is not a problem per se for proving completeness as completeness is only about exhibiting one particular model and it is possible to ensure that $\dot\neg\dot\neg X \dot\Rightarrow X$ holds in this particular model. However, whether the metalanguage is classical or not has an impact on the soundness property, i.e. on the statement that the provability of A implies the validity of A. Indeed, there is little hope to prove the soundness of double-negation elimination if the quantification over models include non-classical models. Therefore, for the definition of validity to be both sound and complete for classical provability with respect to Tarski semantics, independently of whether the metalanguage is intuitionistic or classical, we would need to define classical validity using an explicit restriction to classical models:

$$\mathcal{T} \vDash A \triangleq \forall \mathcal{M}\, \forall \sigma\, (\mathcal{M} \vDash_\sigma \mathit{Classic} \Rightarrow \mathcal{M} \vDash_\sigma \mathcal{T} \Rightarrow \mathcal{M} \vDash_\sigma A)$$

- *Bivalent semantics (predicates as binary functions).* Another approach is to assign to formulae a truth value in the two-valued set $\{0, 1\}$ and to define $\mathcal{M} \vDash_\sigma A$ as $\mathit{truth}_{\mathcal{M}}(A, \sigma) = 1$ for the corresponding *truth* function. This is the approach followed e.g. in Rasiowa-Sikorski's proof, or also e.g. in [15, 76], among others[6]. In particular, relying on a two-valued truth makes the theory *Classic* automatically true.

  Depending on the metalanguage, a function from $\mathcal{F}orm$ to $\{0, 1\}$ can itself be represented either as a functional relation, i.e. as a relation *istrue* on $\mathcal{F}orm \times \{0, 1\}$ such that for all A, there is a unique b such that $\mathit{istrue}(A, b)$ holds (this is the representation used e.g. for the completeness proof in [76]), or, primitively as a function if ever the metalanguage provides such primitive notion of function (as is typically the case in intuitionistic logics, e.g. Heyting Arithmetic in finite types [80], or Martin-Löf's type theory [68, 21]).

  Reverse mathematics of the subsystems of classical second-order arithmetic have shown that building a model from a proof of consistency requires the full strength of Σ_1^0-separability, or equivalently, of Weak Kőnig's Lemma [76]. This implies that the corresponding *truth* function is in general not recursive [58]. Expecting *truth* to be definable primitively as a computable function in an intuitionistic logic is thus hopeless. As for representing truth by a functional relation *istrue*, the expected property $\mathit{istrue}(A, 0) \vee \mathit{istrue}(A, 1)$ could only be proven by requiring some amount of classical reasoning.

  It is known how to compute with classical logic in second-order arithmetic [37, 71, 64] and we could study the computational content of a formalisation of the completeness proof which uses this definition of truth. The extra need for classical reasoning in this approach looks however like a useless complication, so we shall concentrate on the predicates-as-predicates approach.

[5]For instance, in second-order intuitionistic arithmetic extended with Church Thesis (CT), excluded-middle on undecidable formulae is provably contradictory and the same model interpreting X as $\mathit{coh}_{ML} \vee \neg \mathit{coh}_{ML}$ invalidates $\dot\neg\dot\neg X \dot\Rightarrow X$.

[6]For instance, the definition of validity used in Henkin [39], though not fully formal, also intends a two-valued semantics.

So, to summarise, we will not expect truth to be two-valued and will require explicitly as a counterpart that models are classical, leading to the following refined definitions[7] of validity and existence of a model:

$$\begin{array}{lcl} \mathcal{T} \vDash A & \triangleq & \forall \mathcal{M} \forall \sigma\, (\mathcal{M} \vDash_\sigma \mathit{Classic} \Rightarrow \mathcal{M} \vDash_\sigma \mathcal{T} \Rightarrow \mathcal{M} \vDash_\sigma A) \\ \mathcal{T} \text{ has a model} & \triangleq & \exists \mathcal{M} \exists \sigma\, (\mathcal{M} \vDash_\sigma \mathit{Classic} \wedge \mathcal{M} \vDash_\sigma \mathcal{T}) \end{array}$$

Two auxiliary choices of presentation of Tarski semantics can be made[8].

- *Recursively-defined truth.* The approach followed e.g. in the Handbook of Mathematical Logic [6] or the Handbook of Proof Theory [14] is to have the model interpret only the predicate symbols and to have the truth of formulae defined recursively. This is obtained by giving an interpretation $\mathcal{P}$ where any symbol $P \in \mathcal{P}\mathit{red}$ is mapped to a set $\mathcal{P}(P) \subset \mathcal{D}^{a_P}$. Then, the truth of a formula with respect to some assignment σ of the free variables is given recursively by:

$$\begin{array}{lcl} \mathcal{M} \vDash_\sigma P(t_1, \ldots, t_{a_f}) & \triangleq & ([\![t_1]\!]^\sigma_\mathcal{M}, \ldots, [\![t_{a_P}]\!]^\sigma_\mathcal{M}) \in \mathcal{P}(P) \\ \mathcal{M} \vDash_\sigma \dot\bot & \triangleq & \bot \\ \mathcal{M} \vDash_\sigma A \dot\Rightarrow B & \triangleq & \mathcal{M} \vDash_\sigma A \Rightarrow \mathcal{M} \vDash_\sigma B \\ \mathcal{M} \vDash_\sigma \dot\forall x\, A & \triangleq & \forall v \in \mathcal{D}\, \mathcal{M} \vDash_{\sigma \cup [x \leftarrow v]} A \end{array}$$

- *Axiomatically-defined truth.* A common alternative approach is to define truth as a subset $\mathcal{S}$ of closed formulae in the language of terms extended with the constants of $\mathcal{D}$, such that: $\bot$ is not in $\mathcal{S}$; $A \dot\Rightarrow B$ is in $\mathcal{S}$ iff B is whenever A is; $\dot\forall x\, A$ is in $\mathcal{S}$ iff $A[x \leftarrow v]$ is for all values $v \in \mathcal{D}$; $A[x \leftarrow f(v_1, ..., v_n)]$ is in $\mathcal{S}$ iff $A[x \leftarrow v]$ is in $\mathcal{S}$ whenever $\mathcal{F}(f)(v_1, ..., v_n) = v$ for some value $v \in \mathcal{S}$. This approach is adopted e.g. in Krivine [65].

We will retain the first approach which conveniently exempts us from defining the set of formulae enriched with constants from $\mathcal{D}$. So, shortly, a model $\mathcal{M}$ will be a triple $(\mathcal{D}, \mathcal{F}, \mathcal{P})$ where $\mathcal{F}$ maps any symbol $f \in \mathcal{F}\mathit{un}$ to a function $\mathcal{F}(f) \in \mathcal{D}^{a_f} \to \mathcal{D}$ and $\mathcal{P}$ maps any symbol $P \in \mathcal{P}\mathit{red}$ to a set $\mathcal{P}(P) \subset \mathcal{D}^{a_P}$.

1.5. Regarding the metalanguage as a formal system. Let M be the metalanguage in which completeness is stated and O be the object language used to represent provability in first-order logic. In M, a proof of the validity of a formula A is essentially a proof of the universal closure of A, seen as a formula of M, with the closure made over the domain of quantification of quantifiers, over the free predicate symbols, over the free function symbols and over the free variables of A. Otherwise said, adopting a constructive view at proofs of the metalanguage, we can think of the weak completeness theorem in form C1 as a process to transform a proof of the universal closure of A expressed in M into a proof of A expressed in the proof object language O (and conversely,

[7] For the record, note that, in the presence of only negative connectives, an equivalent way to define $\vDash A$ so that it means the same in an intuitionistic and classical setting is to replace the definition of $\mathcal{M} \vDash_\sigma P(t_1, ..., t_{a_P})$ by

$$\mathcal{M} \vDash_\sigma P(t_1, ..., t_{a_P}) \triangleq \neg\neg([\![t_1]\!]^\sigma_\mathcal{M}, \ldots, [\![t_{a_P}]\!]^\sigma_\mathcal{M}) \in \mathcal{M}(P)$$

or even, saving a negation as in Krivine [62], by

$$\mathcal{M} \vDash_\sigma P(t_1, ..., t_{a_P}) \triangleq \neg([\![t_1]\!]^\sigma_\mathcal{M}, \ldots, [\![t_{a_P}]\!]^\sigma_\mathcal{M}) \in \mathcal{M}(P)$$

Indeed, in these cases, the definition of truth becomes a purely negative formula for which intuitionistic and classical provability coincide.

[8] These auxiliary choices would have been relevant as well if we had chosen to represent truth as a map to $\{0, 1\}$.

the soundness theorem can be seen as stating an embedding of O into M). Similarly, a proof of the validity of a formula A with respect to an infinite theory $\mathcal{T}$ is a proof in M of the universal closure of $(\forall B \in \mathcal{T} [\![B]\!]) \Rightarrow [\![A]\!]$ where $[\![C]\!]$ is the replication of C as a formula of M and, computationally speaking, statement S1 is a process to turn such a proof in M (which has to use only a finite subset of $\mathcal{T}$ in M, since M, seen itself as a formal system, supports only finite proofs) into a proof in O.

The key point is however that this transformation of a proof in M into a proof in O is done in M itself, and, within M itself, the only way to extract information out of a proof of validity is by instantiating the free symbols of the interpretation of A in M by actual function and predicate symbols of M, i.e. by producing what at the end is a model, i.e. a domain, functions and predicates actually definable in M.

1.6. Former results about the computational content of completeness proofs for intuitionistic logic. It is known that composing the soundness and completeness theorems for propositional or predicate logic gives a cut-elimination theorem, as soon as completeness is formulated in such a way that it produces a normal proof[9]. Now, if the proofs of soundness and completeness are formalised in a metalanguage equipped with a normalisation procedure, e.g. in a λ-calculus-based proofs-as-programs presentation of second-order arithmetic [61, 63, Ch. 9], one gets an effective cut-elimination theorem, namely an effective procedure which turns any non-necessarily-normal proof of $\Gamma \vdash A$ into a normal proof of $\Gamma \vdash A$.

In the context of intuitionistic provability, this has been explored abundantly under the name of *semantic normalisation*, or *normalisation by evaluation*. Initially based on ideas from Berger and Schwichtenberg [10] in the context of simply-typed λ-calculus, it was studied for the realisability semantics of second-order implicative propositional logic[10] by Altenkirch, Hofmann and Streicher [3], for the realisability semantics of implicative propositional logic in Hilbert style[11] by Coquand and Dybjer [20], for the Kripke semantics of implicative propositional logic in natural deduction style[12] by C. Coquand [19], for Heyting algebras by Hermant and Lipton [43, 44], etc. It has also been applied to phase semantics of linear logic by Okada [70]. It also connects to a normalisation technique in computer science called Typed-Directed Partial Evaluation (TDPE) [23].

Let us recall how this approach works in the case of minimal implicative propositional logic (Figure 1) using soundness and completeness with respect to Kripke models [19]. Let $\mathcal{K}$ range over Kripke models $(\mathcal{W}, \leq, \Vdash_X)$ where $\leq$ is a preorder on $\mathcal{W}$ and $\Vdash_X$ a monotonic predicate over $\mathcal{W}$ for each propositional atom X. Let w range over $\mathcal{W}$, i.e. worlds in the corresponding Kripke models. Let us write $w \Vdash_{\mathcal{K}} A$ (resp. $w \Vdash_{\mathcal{K}} \Gamma$) for truth of A (resp. for the conjunction of the truth of all formulae in Γ) at world w in the Kripke model $\mathcal{K}$. In particular, $w \Vdash_{\mathcal{K}} A$ is extended from atoms to all formulae by defining $w \Vdash_{\mathcal{K}} A \Rightarrow B \triangleq \forall w'(w' \geq w \Rightarrow w' \Vdash_{\mathcal{K}} A \Rightarrow w' \Vdash_{\mathcal{K}} B)$. Let us write $\Gamma \vDash_I A$ for the validity of A relative to Γ at all worlds of all Kripke models, i.e. for the formula $\forall \mathcal{K} \forall w\, (w \Vdash_{\mathcal{K}} \Gamma \Rightarrow w \Vdash_{\mathcal{K}} A)$.

The metalanguage being here a λ-calculus, we shall write its proofs as mathematical functions. We write $x \mapsto t$ for the proof of an implication as well as for the proof of a

[9]Using e.g. Beth-Hintikka-Kanger-Schütte's proof.
[10]I.e., Girard-Reynolds System F.
[11]I.e., equivalently, simply-typed combinatory logic.
[12]I.e., equivalently, simply-typed λ-calculus.

universal quantification, possibly also writing $(x : A) \mapsto t$ to make explicit that x is the name of a proof of A. We shall represent modus ponens and instantiation of universal quantification by function application, written $t\,u$. We shall use the notation $()$ for the canonical proof of a nullary conjunction and the notation (t, u) for the proof of a binary conjunction, seen as a product type and obtained by taking the pair of the proofs of the components of the conjunction. To give a name f to the proof of a statement of the form $\forall x_1, ..., x_n\,(A \Rightarrow B)$ we shall use the notation $f^{x_1,...,x_n}(a : A) : B$ followed by clauses of the form $f^{x_1,...,x_n}(a) \triangleq t$ (for readability, we may also write some of the x_i as subscripts rather than superscripts of f).

For instance, the proof that Kripke forcing is monotone, i.e. that $\forall ww'\,(w' \geq w \wedge w \Vdash A \Rightarrow w' \Vdash A)$, can be written as the following function $\Uparrow_A$, recursive in the structure of A, taking as arguments two worlds w and w':

$$\begin{array}{llllll}
\Uparrow_A^{w,w'} & : & w' \geq w \;\wedge\; w \Vdash A & & \Rightarrow & w' \Vdash A \\
\Uparrow_X^{w,w'} & & (\,h \qquad ,\; m &) & \triangleq & p_X(h, m) \\
\Uparrow_{A\Rightarrow B}^{w,w'} & & (\,h \qquad ,\; m &) & \triangleq & w'' \mapsto (h' : w'' \geq w') \mapsto m\,w''\,(trans(h, h'))
\end{array}$$

where p_X is the proof of monotonicity of $\Vdash_X$ and *trans* is the proof of transitivity of $\geq$, both coming with the definition of Kripke models, while, in the definition, h is a proof of $w' \geq w$ and m a proof of $w \Vdash A$.

Similarly, the extension of $\Uparrow$ to a proof that forcing of contexts is monotone can be written as follows, where we reuse the notation $\Uparrow$, now with a context as index, to denote a proof of $\forall ww'\,(w' \geq w \wedge w \Vdash \Gamma \Rightarrow w' \Vdash \Gamma)$:

$$\begin{array}{llllll}
\Uparrow_\Gamma^{w,w'} & : & w' \geq w \;\wedge\; w \Vdash \Gamma & & \Rightarrow & w' \Vdash \Gamma \\
\Uparrow_\epsilon^{w,w'} & & (\,h \qquad ,\; () &) & \triangleq & () \\
\Uparrow_{\Gamma,A}^{w,w'} & & (\,h \qquad ,\; (\sigma, m) &) & \triangleq & (\Uparrow_\Gamma^{w,w'}(h, \sigma), \Uparrow_A^{w,w'}(h, m))
\end{array}$$

Let us write $\Gamma \vdash_I A$ for intuitionistic provability. Let us consider the canonical proof $soundness_A^\Gamma$ of $(\Gamma \vdash_I A) \Rightarrow (\Gamma \Vdash_I A)$ proved by induction on the derivation of $\Gamma \vdash_I A$. We write the proof as a recursive function, recursively on the structure of formulae:

$$\begin{array}{llll}
soundness_A^\Gamma \;: & \Gamma \vdash_I A & \Rightarrow & \Gamma \models_I A \\
soundness_A^\Gamma & \dot{ax}_i & \triangleq & \mathcal{K} \mapsto w \mapsto \sigma \mapsto \sigma(i) \\
soundness_{A\Rightarrow B}^\Gamma & \dot{abs}(p) & \triangleq & \mathcal{K} \mapsto w \mapsto \sigma \mapsto w' \mapsto (h : w \leq w') \mapsto m \mapsto \\
 & & & soundness_B^{\Gamma,A}\,p\,\mathcal{K}\,w'\,(\Uparrow_\Gamma^{w,w'}(h, \sigma), m) \\
soundness_B^\Gamma & \dot{app}(p, q) & \triangleq & \mathcal{K} \mapsto w \mapsto \sigma \mapsto \\
 & & & (soundness_{A\Rightarrow B}^\Gamma\,p\,\mathcal{K}\,w\,\sigma)\,w\,refl\,(soundness_A^\Gamma\,q\,\mathcal{K}\,w\,\sigma)
\end{array}$$

where u is a proof of $\Gamma \vdash_I A$ in the last line, $\dot{app}$, $\dot{abs}$, $\dot{ax}_i$ are the name of inference rules defining object-level implicational propositional logic in a natural deduction style (see Figure 1); $\sigma(i)$ is the $(i+1)^{\text{th}}$ component of σ starting from the right, and *refl* is the proof of reflexivity of $\geq$ coming with the definition of Kripke models.

Let us also consider the following somehow canonical proof of cut-free completeness, *completeness* : $(\Gamma \models_I A) \Rightarrow (\Gamma \vdash_I^{cf} A)$. It is based on the universal model of context $\mathcal{K}_0$ defined by taking for $\mathcal{W}$ the set of contexts Γ ordered by inclusion and $\Gamma \vdash_I^{cf} X$ for the forcing $\Vdash_X$ of atom X at world Γ. Now, the proof proceeds by showing the two directions of $\Gamma \Vdash_{\mathcal{K}_0} A \Leftrightarrow \Gamma \vdash_I^{cf} A$ by mutual induction on A. It is common to write $\downarrow$ for the left-to-right direction (called *reify*, or *quote*) and $\uparrow$ for the right-to-left direction

Primitive rules

$$\frac{|\Gamma'| = i}{\Gamma, A, \Gamma' \vdash_I A}\ \dot{ax}_i \qquad \frac{\Gamma \vdash_I A \dot{\Rightarrow} B \qquad \Gamma \vdash_I A}{\Gamma \vdash_I B}\ \dot{app} \qquad \frac{\Gamma, A \vdash_I B}{\Gamma \vdash_I A \dot{\Rightarrow} B}\ \dot{abs}$$

Admissible rule

$$\frac{\Gamma \subset \Gamma' \qquad \Gamma \vdash_I A}{\Gamma' \vdash_I A}\ \dot{weak}$$

FIGURE 1. Inference rules characterising minimal implicational logic

(called *reflect*, or *eval*):

$$\begin{array}{llcl}
\downarrow^{\Gamma}_{A} : & \Gamma \Vdash_{\mathcal{K}_0} A & \Rightarrow & \Gamma \vdash^{cf}_{I} A \\
\downarrow^{\Gamma}_{P} & m & \triangleq & m \\
\downarrow^{\Gamma}_{A \dot{\Rightarrow} B} & m & \triangleq & \dot{abs}\,(\downarrow^{\Gamma,A}_{B}\ (m\ (\Gamma, A)\ inj^{A}_{\Gamma}\ (\uparrow^{\Gamma,A}_{A}\ \dot{ax}_0)))
\end{array}$$

$$\begin{array}{llcl}
\uparrow^{\Gamma}_{A} : & \Gamma \vdash^{cf}_{I} A & \Rightarrow & \Gamma \Vdash_{\mathcal{K}_0} A \\
\uparrow^{\Gamma}_{P} & p & \triangleq & p \\
\uparrow^{\Gamma}_{A \dot{\Rightarrow} B} & p & \triangleq & \Gamma' \mapsto f \mapsto m \mapsto \uparrow^{\Gamma'}_{B}\ (\dot{app}(\dot{weak}(f, p), \downarrow^{\Gamma'}_{A}\ m))
\end{array}$$

$$\begin{array}{lcl}
init^{\Gamma}_{\Gamma'} & : & \Gamma \Vdash_{\mathcal{K}_0} \Gamma' \\
init^{\Gamma}_{\epsilon} & \triangleq & () \\
init^{\Gamma}_{\Gamma',A} & \triangleq & (init^{\Gamma}_{\Gamma'}, \uparrow^{\Gamma}_{A}\ (\dot{ax}_{|\Gamma|-|\Gamma',A|}))
\end{array}$$

$$\begin{array}{llcl}
completeness^{\Gamma}_{A} : & \Gamma \vDash_I A & \Rightarrow & \Gamma \vdash^{cf}_{I} A \\
completeness^{\Gamma}_{A} & m & \triangleq & \downarrow^{\Gamma}_{A}\ (m\ \mathcal{K}_0\ \Gamma\ init^{\Gamma}_{\Gamma})
\end{array}$$

where $|\Gamma|$ is the length of Γ, $\dot{weak}$ is the admissible rule of weakening in object-level implicational propositional logic and inj^{A}_{Γ} is a proof of $\Gamma \subset \Gamma, A$.

In particular, by placing ourselves in a metametalanguage, such that the metalanguage is seen as a proofs-as-programs-style natural deduction object language, i.e. as a typed λ-calculus, one would be able to show that

- for every given proof of $\Gamma \vdash_I A$, soundness produces, by normalisation of the metalanguage[13], a proof of $\Gamma \vDash_I A$ whose structure follows the one of the proof of $\Gamma \vdash_I A$;
- for every proof of validity taken in canonical form (i.e. as a closed β-normal η-long λ-term of type $\Gamma \vDash A$ in the metalanguage), the resulting proof of $\Gamma \vdash^{cf}_{I} A$ obtained by completeness is, by normalisation in the metalanguage, the same λ-term with the abstractions and applications over $\mathcal{K}$, w and proofs of $w \leq w'$ removed.

On the other side, if our proofs-as-programs-based metalanguage is able to state properties of its proofs (as is the case for instance of Martin-Löf's style type theories [68]),

[13]Typically proved by embedding in another language assumed to be consistent.

it can be shown within the metalanguage itself that the composition of completeness and soundness produces normal forms. This is what C. Coquand did by showing that the above proofs of soundness and completeness, seen as typed programs, satisfy the following properties:

$$\forall p : (\Gamma \vdash_I A)\ p \sim \mathit{soundness}_A^\Gamma\ p\, \mathcal{K}_0\, \Gamma\, \mathit{init}_\Gamma^\Gamma$$
$$\forall p : (\Gamma \vdash_I A)\, \forall m : (\Gamma \vDash_I A)\ (p \sim m\, \mathcal{K}_0\, \Gamma\, \mathit{init}_\Gamma^\Gamma \ \Rightarrow\ p =_{\beta\eta} \mathit{completeness}_A^\Gamma\, m)$$

where $\sim$ is an appropriate "Tait computability" relation between object proofs and semantic proofs expressing that *soundness p* reflects p.

Then, since *completeness* returns normal forms, we get that the composite function *completeness* (*soundness p*) evaluates to a normal form q such that $q =_\beta p$.

Let us conclude this section by saying that the extension of this proof to universal quantification and falsity, using so-called exploding nodes, has been studied e.g. in [42]. The extension to first-order classical logic has been studied e.g. in [52]. The case of disjunction and existential quantification is typically addressed using variants of Kripke semantics [51], Beth models, topological models [22], or various alternative semantics (e.g. [1, 2, 70, 73]).

One of the purposes of this paper is precisely to start a comparative exploration of the computational contents of proofs of Gödel's completeness theorem and of the question of whether they provide a normalisation procedure. In the case of Henkin's proof, the answer is negative: even if the resulting object-level proof that will be constructively obtained in Section 2 is related to the proof of validity in the meta-logic, it is neither cut-free nor isomorphic to it. In particular, it drops information from the validity proof by sharing subparts that prove the same subformula as will be emphasised in Section 2.2.

1.7. The intuitionistic provability of the different statements of completeness. Statements C1, C2 and C3, as well as statements S1, S2, S3 are classically equivalent but not intuitionistically equivalent. In particular, only C2 and S2 are intuitionistically provable.

More precisely, since our object language has only negative connectives, the formula $\mathcal{M} \vDash_\sigma \mathcal{T}$ is in turn composed of only negative connectives in the metalanguage. Hence, the only positive connective in the statements C2 and S2 is the existential quantifier asserting the existence of a model.

This existential quantifier is intuitionistically provable as our formulation of Henkin's proof of S2 given in the next section shows: given a proof of consistency of a theory, we can produce a syntactic model in the form of a specific predicate. It shall however be noted that this predicate is not itself recursive in general, since constructing this model is in general equivalent to producing an infinite path in any arbitrary infinite binary tree (such an infinite path is a priori not recursive, see Kleene [58], Simpson [76]).

Otherwise, from an intuitionistic point of view, statements C1 and S1 are particularly interesting, as they promise to produce (object) proofs in the object language out of proofs of validity in the metalanguage. However, Kreisel [59] showed, using a result by Gödel [36], that C1 is equivalent to Markov's principle over intuitionistic second-order arithmetic. This has been studied in depth by McCarty [69] ant it turns out that S1 is actually equivalent to Markov's principle if the theory is recursively enumerable.

However, for arbitrary theories, reasoning by contradiction on formulae of arbitrarily large logical complexity is in general needed as the following adaptation of McCarty's proof shows: Let $\mathcal{A}$ be an arbitrary formula of the metalanguage and consider e.g. the theory defined by $B \in \mathcal{T} \triangleq (B = \bot) \wedge \mathcal{A} \vee (B = X) \wedge \neg\mathcal{A}$ for X a distinct propositional atom of the object language. We intuitionistically have that $\mathcal{T} \vDash X$ because this is a negative formulation of a classically provable statement[14]. By completeness, we get $\mathcal{T} \vdash X$, and, by case analysis on the normal form of the so-obtained proof, we infer that either $\mathcal{A}$ or $\neg\mathcal{A}$.

The need for Markov's principle is connected to how $\bot$ is interpreted in the model. Krivine [65] showed that for a language without $\bot$[15], C1 is provable intuitionistically. As analysed by Berardi [7] and Berardi and Valentini [8], Markov's principle is not needed anymore if we additionally accept the degenerate model where all formulae including $\bot$ are interpreted as true[16]. Let us formalise this precisely.

We define a *possibly-exploding* model $\mathcal{M}$ to be a model $(\mathcal{D}, \mathcal{F}, \mathcal{P}, \mathcal{A}_\bot)$ such that $(\mathcal{D}, \mathcal{F}, \mathcal{P})$ is a model in the previous sense and $\mathcal{A}_\bot$ a fixed formula intended to interpret $\bot$. The definition of truth is then modified as follows:

$$\mathcal{M} \vDash^e_\sigma \bot \quad \triangleq \quad \mathcal{A}_\bot$$

with the rest of clauses unchanged[17]

Note that because $\bot \dot{\Rightarrow} A$ is a consequence of $\dot{\neg}\dot{\neg}A \dot{\Rightarrow} A$, the following holds for all A and all σ in any classical possibly-exploding model:

$$\mathcal{A}_\bot \;\Rightarrow\; \mathcal{M} \vDash^e_\sigma A\,,$$

so we do not need to further enforce it[18]. Let us rephrase C1 and S1 using classical possibly-exploding models:

---

[14]Let $\mathcal{M}$ and σ such that $\mathcal{M} \vDash_\sigma \mathcal{T}$. We first show $\neg\mathcal{A}$. Indeed, if $\mathcal{A}$ holds, then $\bot \in \mathcal{T}$ and we get by $\mathcal{M} \vDash_\sigma \mathcal{T}$ that the model is contradictory. But if $\neg\mathcal{A}$, then $X \in \mathcal{T}$, hence $\mathcal{M} \vDash_\sigma X$.

[15]So-called minimal classical logic in [4], which is however not functionally complete since no formula can represent the always-false function.

[16]This is similar to the approach followed by Friedman [31] and Veldman [82] to intuitionistically prove the completeness of intuitionistic logic with respect to a relaxing of Beth models with so-called fallible models, and to a relaxing of Kripke models with so-called exploding nodes, respectively.

[17]In [8], a classical possibly-exploding model is called a minimal model, in reference to minimal logic [55]. The difference between non-exploding models and possibly-exploding models can actually be interpreted from the point of view of linear logic not as a difference of definition of models but as a difference of interpretations of the false connective. An non-exploding model is a model where the false connective is interpreted as the connective 0 of linear logic (a positive connective, neutral for the standard disjunction and with no introduction rule). A possibly-exploding model is a model where the false connective is interpreted as the connective $\bot$ of linear logic (a negative connective informally standing for an empty sequent). See e.g. Okada [70] or Sambin [73] for examples of differences of interpretation of 0 and $\bot$ in completeness proofs for linear logic.

[18]As a matter of purity, since it is standard that the classical scheme $\dot{\neg}\dot{\neg}A \dot{\Rightarrow} A$ is equivalent to the conjunction of a purely classical part, namely Peirce's law representing the scheme $((A \dot{\Rightarrow} B) \dot{\Rightarrow} A) \dot{\Rightarrow} B$ and of a purely intuitionistic part, namely ex falso quodlibet representing the scheme $\bot \dot{\Rightarrow} A$, we could have decomposed *Classic* into the union of $\mathcal{P}eirce \triangleq \{((A \dot{\Rightarrow} B) \dot{\Rightarrow} A) \dot{\Rightarrow} A \mid A \in \mathcal{F}orm\}$ and of $\mathcal{E}xfalso \triangleq \{\bot \dot{\Rightarrow} A \mid A \in \mathcal{F}orm\}$.

As already said in Section 1.4, the condition $\mathcal{M} \vDash_\sigma$ *Classic*, and in particular the conditions $\mathcal{M} \vDash_\sigma \mathcal{P}eirce$ and $\mathcal{M} \vDash_\sigma \mathcal{E}xfalso$ are needed to show soundness with respect to classical models in a minimal setting. In an intuitionistic setting, $\mathcal{M} \vDash_\sigma \mathcal{E}xfalso$ holds by default and does not have to be explicitly enforced. In a classical setting, $\mathcal{M} \vDash_\sigma \mathcal{P}eirce$ does not have to be explicitly enforced. So, requiring these conditions is to ensure that the definition of validity is the one we want independently of the specific properties of the metalanguage.

In contrast, for the purpose of completeness, possibly-exploding models are needed for an intuitionistic proof of C1 to be possible, but none of $\mathcal{P}eirce(\mathcal{M})$ and $\mathcal{E}xfalso(\mathcal{M})$ are required.

$$C1'. \quad \models^e A \Rightarrow \vdash A$$
$$S1'. \quad \mathcal{T} \models^e A \Rightarrow \mathcal{T} \vdash A$$

where

$$\mathcal{T} \models^e A \triangleq \forall \mathcal{M} \forall \sigma (\mathcal{M} \models^e_\sigma \mathit{Classic} \Rightarrow \mathcal{M} \models^e_\sigma \mathcal{T} \Rightarrow \mathcal{M} \models^e_\sigma A)$$

In particular, it is worthwhile to notice that $\mathcal{T} \models^e A$ and $\mathcal{T} \models A$ are classically equivalent since, up to logical equivalence, $\models^e$ only differs from $\models$ by an extra quantification over the degenerate always-true model. Hence C1 and C1', as well as S1 and S1', are classically equivalent too. But C1' as well as S1' are intuitionistically provable for recursively enumerable theories, while C1 and S1, even for recursively enumerable theories, would require Markov's principle[19].

Let us conclude this section by considering C3 and S3. These statements are not intuitionistically provable: if they were, provability could be decided. This does not mean however that we cannot compute with C3 and S3. Classical logic is computational (see e.g. [71]), but for an evaluation to be possible, an interaction with a proof of a statement which uses C3 or S3 is needed. The formalisation by Blanchette, Popescu and Traytel [12] might be a good starting point to analyse the proof but we will not explore this further here.

1.8. Chronology and recent related works. The extraction of a computational content from Henkin's proof was obtained by the authors from an analysis of the formalisation [49] in the Coq proof assistant [17] of Henkin's proof. It was first presented at the TYPES conference in Warsaw, 2010. The paper was essentially written in 2013 but it remained in draft and unstable form until 2016. A non-peer-reviewed version was made publicly available on the web page of the first author late 2016 and slightly updated in 2019. Our constructive presentation of Henkin's proof inspired Forster, Kirst and Wehr to formalise the completeness theorem in the intuitionistic setting of Coq [27, 28]. This urged us to polish the paper one step further and to submit it for publication.

The need for a weakening rule was not addressed in the 2016 version. This is fixed in the current version which at the same time presents the proof in a more structured way. The extension of our intuitionistic version of Henkin's proof to the disjunction is also new. It relies on (a slight generalisation of) the weakly classical principle of Double Negation Shift (DNS) which is known to preserve the witness and disjunction properties of intuitionistic logic.

Various investigations of Gödel's completeness in an intuitionistic setting have been published since our proof was first presented. To our knowledge, in addition to [27, 28], this includes papers by Krivtsov [67] in intuitionistic arithmetic[20] and Espíndola [24, 25] in intuitionistic set theory. In particular, Krivtsov showed that Gödel's completeness with respect to exploding models is equivalent to the Weak Fan Theorem, which is an intuitionistic counterpart to the equivalence of Gödel's completeness with Weak Kőnig's Lemma in classical reverse mathematics of the subsystems of second-order arithmetic [76]. There are however different variants of the Weak Fan Theorem (WFT) depending on how infinite paths are represented in a binary tree (see [13]). Let us call

[19]Markov's principle can actually be "intuitionistically" implemented e.g. by using an exception mechanism [41], so a computational content to weak completeness and strong completeness for recursively enumerable theories could as well be obtained without any change to the interpretation of $\bot$.

[20]Krivtsov gives an intuitionistic proof of the Beth-Hintikka-Kanger-Schütte style

WFT_{pred} the intuitionistically provable[21] variant of WFT where infinite paths are represented using a predicate and WFT_{dec} the weakly classical variant[22] where infinite paths are represented using a decidable predicate. We conjecture, consistently with the equivalence between WFT_{pred} and completeness with respect to Scott entailment relations in [13] that Krivtsov's result can be strengthened into an equivalence between exploding completeness without disjunction nor existential quantification and WFT_{pred} while, as in Krivtsov, exploding completeness in the presence of disjunction and existential completeness is equivalent to WFT_{dec}. In turn, we conjecture that WFT_{dec}, thus completeness with respect to exploding models and all connectives, is equivalent to WFT_{pred} together with the slight generalisation of DNS used in Section 2.6.

§2. The computational content of Henkin's proof of Gödel's completeness. We now recall Henkin's proof of completeness and analyse its computational content.

2.1. Henkin's proof of statement S2, slightly simplified. We give a simplified form of Henkin's proof of the strong form of Gödel's completeness theorem [39], formulated as statement S2. The simplification is on the use of free variables instead of constants in Henkin axioms and in the use of only implicative formulae in the process of completion of a consistent set of formulae into a maximally consistent one[23].

Let $\mathcal{T}$ be a consistent set of formulae mentioning an at most countable[24] number of function symbols and predicate symbols. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be countable sets of variables forming a partition of $\mathcal{X}$. We can assume without loss of generality that the free variables of the formulae in $\mathcal{T}$ are in $\mathcal{X}_1$ leaving $\mathcal{X}_2$ as a pool of fresh variables.

We want to show that $\mathcal{T}$ has a model, and for that purpose, we shall complete it into a consistent set $\mathcal{S}_\omega$ of formulae which is maximal in the sense that if $A \notin \mathcal{S}_\omega$ then $\dot\neg A \in \mathcal{S}_\omega$. We shall also ensure that for every universally quantified formula $\dot\forall x\, A(x)$, there is a corresponding so-called Henkin axiom $A(y) \mathrel{\dot\Rightarrow} \dot\forall x\, A(x)$ in $\mathcal{S}_\omega$ with y fresh in $\dot\forall x\, A(x)$. For the purpose of this construction, we fix an injective enumeration ϕ of formulae of the form $\dot\forall x\, A(x)$ or $A \mathrel{\dot\Rightarrow} B$ and write $\lceil A \rceil$ for the index of a formula A of such form in the enumeration. We also take ϕ so that formulae of even index are of the form $\dot\forall x\, A(x)$ and formulae of odd index are of the form $A \mathrel{\dot\Rightarrow} B$.

Let $\mathcal{S}_0$ be $\mathcal{T}$ and assume that we have already built $\mathcal{S}_n$. If n is even, $\phi(n)$ has the form $\dot\forall x\, A(x)$. We then consider a variable $x_{n/2} \in \mathcal{X}_2$ which is fresh in all $\phi(i)$ for $i \le n$ and we set $\mathcal{S}_{n+1} \triangleq \mathcal{S}_n \cup (A(x_{n/2}) \mathrel{\dot\Rightarrow} \dot\forall x\, A(x))$. Otherwise, $\phi(n)$ is an implicative formula and we consider two cases. If $\mathcal{S}_n \cup \phi(n)$ is consistent, i.e., if $(\mathcal{S}_n \cup \phi(n) \vdash \dot\bot) \Rightarrow \bot$, we set $\mathcal{S}_{n+1} \triangleq \mathcal{S}_n \cup \phi(n)$. Otherwise, we set $\mathcal{S}_{n+1} \triangleq \mathcal{S}_n$. We finally define the predicate $A \in \mathcal{S}_\omega \triangleq \exists n\, (\mathcal{S}_n \vdash A)$, i.e. $\exists n \exists \Gamma \subset \mathcal{S}_n\, (\Gamma \vdash A)$, and this is the basis of a syntactic model $\mathcal{M}_0$ defined by taking

[21] See a proof relying on intuitionistic ACA_0 in the Coq standard library [18, WeakFan.v]

[22] See Berger [9] who identifies the classical part of (a functional form of) WFT as a principle called L_{fan}.

[23] Smullyan [77] credits Hasenjaeger [38] and Henkin independently for proof variants using free variables (see also Henkin [40]). In particular, this allows to build a maximal consistent theory in one step instead of a countable number of steps as in Henkin's original proof. See also [76, Th. IV.3.3] for a proof building a maximal consistent theory in one step.

[24] In the presence of uncountably many symbols, one would need the ultrafilter theorem to build the model and this would require extra computational tools to make the proof constructive. See [54] for the equivalence in set theory between the ultrafilter theorem and the completeness theorem on non-necessarily countable signatures.

$$\begin{array}{lcl} \mathcal{D} & \triangleq & \mathcal{T}erm \\ \mathcal{F}(f)(t_1,...,t_n) & \triangleq & f(t_1,...,t_n) \\ \mathcal{P}(P)(t_1,...,t_n) & \triangleq & P(t_1,...,t_n) \in \mathcal{S}_\omega \end{array}$$

By induction, each $\mathcal{S}_n$ is consistent. Indeed, if $\phi(n)$ is implicative and $\mathcal{S}_{n+1} \equiv \mathcal{S}_n \cup \phi(n)$, it is precisely because $\mathcal{S}_{n+1}$ is consistent. Otherwise, the consistency of $\mathcal{S}_{n+1}$ comes from the consistency of $\mathcal{S}_n$. If $\phi(n)$ is some $\dot\forall x\, A(x)$, then $\mathcal{S}_{n+1} \equiv \mathcal{S}_n \cup (A(x_{n/2}) \dot\Rightarrow \dot\forall x\, A(x))$. This is consistent by freshness of $x_{n/2}$ in both $\mathcal{T}$ and in the $\phi(i)$ for $i \le n$. Indeed, because $x_{n/2}$ is fresh, any proof of $\mathcal{S}_n \cup (A(x_{n/2}) \dot\Rightarrow \dot\forall x\, A(x)) \vdash \bot$ can be turned into a proof of $\mathcal{S}_n \cup \dot\neg\dot\forall y\, \dot\neg(A(y) \dot\Rightarrow \dot\forall x\, A(x)) \vdash \bot$, which itself can be turned into a proof of $\mathcal{S}_n \vdash \bot$ since $\dot\neg\dot\forall y\, \dot\neg\, (A(y) \dot\Rightarrow \dot\forall x\, A(x))$ is a classical tautology[25]

Let A be a formula and σ a substitution of its free variables. We now show by induction on the logical depth[26] of A that $\mathcal{M}_0 \models_\sigma A \Leftrightarrow A[\sigma] \in \mathcal{S}_\omega$, where $A[\sigma]$ denotes the result of substituting the free variables of A by the terms in σ. This is sometimes considered an easy combinatoric argument but we shall detail the proof because it is here that the computational content of the proof is non-trivial. Moreover, we do not closely follow Henkin's proof who is making strong use of classical reasoning. We shall instead reason intuitionistically, which does not raise any practical difficulty here.

- Let us focus first on the case when A is $B \dot\Rightarrow C$. One way to show $B[\sigma] \dot\Rightarrow C[\sigma] \in \mathcal{S}_\omega$ from $\mathcal{M}_0 \models_\sigma B \dot\Rightarrow C$ is to show that for n being $\lceil B[\sigma] \dot\Rightarrow C[\sigma] \rceil$, the set $\mathcal{S}_n \cup B[\sigma] \dot\Rightarrow C[\sigma]$ is consistent, i.e. that a contradiction arises from $\mathcal{S}_n \cup B[\sigma] \dot\Rightarrow C[\sigma] \vdash \bot$. Indeed, from the latter, we get both $\mathcal{S}_n \vdash B[\sigma]$ and $\mathcal{S}_n \vdash \dot\neg C[\sigma]$. From $\mathcal{S}_n \vdash B[\sigma]$ we get $\mathcal{M}_0 \models_\sigma B$ by induction hypothesis, hence $\mathcal{M}_0 \models_\sigma C$ by assumption on the truth of $B \dot\Rightarrow C$. Then $C[\sigma] \in \mathcal{S}_\omega$ again by induction hypothesis, hence $\mathcal{S}_{n'} \vdash C[\sigma]$ for some n'. But also $\mathcal{S}_n \vdash \dot\neg C[\sigma]$, hence $\mathcal{S}_{max(n,n')} \vdash \bot$ which contradicts the consistency of $\mathcal{S}_{max(n,n')}$.
- Conversely, if $B[\sigma] \dot\Rightarrow C[\sigma] \in \mathcal{S}_\omega$, this means $\mathcal{S}_n \vdash B[\sigma] \dot\Rightarrow C[\sigma]$ for some n. To prove $\mathcal{M}_0 \models_\sigma B \dot\Rightarrow C$, let us assume $\mathcal{M}_0 \models_\sigma B$. By induction hypothesis we get $\mathcal{S}_{n'} \vdash B[\sigma]$ for some n' and hence $\mathcal{S}_{max(n,n')} \vdash C[\sigma]$, i.e. $C[\sigma] \in \mathcal{S}_\omega$. We conclude by induction hypothesis to get $\mathcal{M}_0 \models_\sigma C$.
- Let us then focus on the case when A is $\dot\forall x\, B$. For n even being $\lceil (\dot\forall x\, B)[\sigma] \rceil$, we have $(B[\sigma, x \leftarrow x_{n/2}] \dot\Rightarrow (\dot\forall x\, B)[\sigma]) \in \mathcal{S}_{n+1}$. From $\mathcal{M}_0 \models_\sigma \dot\forall x\, B(x)$ we get $\mathcal{M}_0 \models_{\sigma, x \leftarrow x_{n/2}} B(x)$ and by the induction hypothesis we then get the existence of some n' such that $\mathcal{S}_{n'} \vdash B[\sigma, x \leftarrow x_{n/2}]$. Hence, $\mathcal{S}_{max(n+1,n')} \vdash (\dot\forall x\, B)[\sigma]$, which means $(\dot\forall x\, B)[\sigma] \in \mathcal{S}_\omega$.
- Conversely, assume $\mathcal{S}_n \vdash (\dot\forall x\, B)[\sigma]$ for some n and prove $\mathcal{M}_0 \models_\sigma \forall x\, B$. Let t be a term. From $\mathcal{S}_n \vdash (\dot\forall x\, B)[\sigma]$ we get $\mathcal{S}_n \vdash B[\sigma, x \leftarrow t]$, hence, by induction hypothesis, $\mathcal{M}_0 \models_{\sigma, x \leftarrow t} B(x)$.
- Let us then consider the case A is $\dot\bot$. By ex falso quodlibet in the metalanguage, it is direct that $\bot \Rightarrow (\dot\bot \in \mathcal{S}_\omega)$.
- Conversely, let us prove $(\dot\bot \in \mathcal{S}_\omega) \Rightarrow \bot$. From $\dot\bot \in \mathcal{S}_\omega$ we know $\mathcal{S}_n \vdash \dot\bot$ for some n which, again, contradicts the consistency of $\mathcal{S}_n$.
- The case when A is $P(t_1, \ldots, t_n)$ is by definition.

[25] The famous "Drinker Paradox".

[26] In particular, we consider $B(t)$ to be smaller to $\forall x\, B(x)$ for any t.

Before completing the proof, it remains to prove that the model is classical. Using the equivalence between $\mathcal{M}_0 \vDash_{id} A$ and $A \in \mathcal{S}_\omega$ for A closed and *id* the empty substitution, it is enough to prove that $\dot\neg\dot\neg A \in \mathcal{S}_\omega$ implies $A \in \mathcal{S}_\omega$. But the former means $\mathcal{S}_n \vdash \dot\neg\dot\neg A$ for some n, hence $\mathcal{S}_n \vdash A$ by classical reasoning in the object language, hence $A \in \mathcal{S}_\omega$.

We are now ready to complete the proof: for every $B \in \mathcal{T}$, since $\mathcal{T} \vdash B$, we get $B \in \mathcal{S}_\omega$ and hence $\mathcal{M}_0 \vDash_{id} B$.

2.2. From Henkin's proof of statement S2 to a proof of statement S1'. Let us fix a formula A_0 and a recursively enumerable theory $\mathcal{T}_0$, i.e. a theory defined by a Σ^0_1-statement. To get a proof of statement S1 for $\mathcal{T}_0$ and A_0 is easy by using Markov's principle: to prove $\mathcal{T}_0 \vdash A_0$ from $\mathcal{T}_0 \vDash A_0$, let us assume the contrary, namely that $\mathcal{T}_0 \cup \dot\neg A_0$ is consistent. Then, we can complete $\mathcal{S}_0 \triangleq \mathcal{T}_0 \cup \dot\neg A_0$ into $\mathcal{S}_\omega$ and build out of it a classical model $\mathcal{M}_0$ such that $\forall B \in \mathcal{T}_0\, \mathcal{M}_0 \vDash_{id} B$ as well as $\mathcal{M}_0 \vDash_{id} \dot\neg A_0$, i.e. $\neg(\mathcal{M}_0 \vDash_{id} A_0)$. But this contradicts $\mathcal{T}_0 \vDash A_0$ and, because $\mathcal{T}_0$ is Σ^0_1, hence $\mathcal{T}_0 \vdash A$ as well, Markov's principle applies.

As discussed in Section 1.7, S1 cannot be proved without Markov's principle, so we shall instead prove S1'. To turn the proof of S2 into a proof of S1' which does not require reasoning by contradiction, we shall slightly change the construction of $\mathcal{S}_\omega$ from $\mathcal{T}_0 \cup \dot\neg A_0$ so that it is not consistent in an absolute sense, but instead consistent relative to $\mathcal{T}_0 \cup \dot\neg A_0$. In particular, we change the condition for extending $\mathcal{S}_{2n+1}$ with $\phi(2n+1)$ to be that $\mathcal{S}_{2n+1} \cup \phi(2n+1)$ is *consistent relative* to $\mathcal{T}_0 \cup \dot\neg A_0$.

Then, we show by induction not that $\mathcal{S}_n$ is consistent but that its inconsistency reduces to the inconsistency of $\mathcal{T}_0 \cup \dot\neg A_0$.

For the construction of the now possibly-exploding model, we take as interpretation of $\bot$ the formula $\mathcal{T}_0, \dot\neg A_0 \vdash \dot\bot$. Proving $\dot\bot \in \mathcal{S}_\omega \Rightarrow \mathcal{M}_0 \vDash^e_\sigma \dot\bot$ now reduces to proving $\mathcal{S}_n \vdash \dot\bot \Rightarrow \mathcal{T}_0, \dot\neg A_0 \vdash \dot\bot$ which is the statement of relative consistency[27]. Conversely, $\mathcal{M}_0 \vDash^e_\sigma \dot\bot \Rightarrow \dot\bot \in \mathcal{S}_\omega$ now comes by definition of $\mathcal{S}_0$.

The change in the definition of $\mathcal{S}_\omega$ as well as the use of possibly-exploding models is connected to Friedman's A-translation [32] being able to turn Markov's principle into an admissible rule. Here, A is the Σ^0_1-formula $\mathcal{T}_0 \vdash A_0$ and by replacing $\bot$ with A in the definition of model, hence of validity, as well as in the definition of $\mathcal{S}_\omega$, we are able to prove $(A \Rightarrow A) \Rightarrow A$ whereas only $(A \Rightarrow \bot) \Rightarrow \bot$ was otherwise provable. Then, A comes trivially from $(A \Rightarrow A) \Rightarrow A$.

This was the idea followed by Krivine [65] in his constructive proof of Gödel's theorem for a language restricted to $\dot\Rightarrow$ and $\dot\forall$, as analysed and clarified in Berardi and Valentini [8].

As a final remark, one could wonder whether the construction of $\mathcal{S}_{2n+2}$ by case on an undecidable statement is compatible with intuitionistic reasoning. Indeed, constructing the sequence of formulae added to $\mathcal{T}_0 \cup \dot\neg A_0$ in order to get $\mathcal{S}_n$ seems to require a use of excluded-middle. However, in the proof of completeness, only the property $A \in \mathcal{S}_n$ matters, and this property is directly definable by induction as

[27] Interestingly enough, since $\mathcal{T}_0 \vdash A$ effectively holds as soon as an effective proof of validity of A is given, the model we build is then the degenerate one in which all formulae are true.

$$
\begin{array}{lcl}
A \in \mathcal{S}_0 & \triangleq & A \in \mathcal{T}_0 \cup \dot{\neg} A_0 \\
A \in \mathcal{S}_{n+1} & \triangleq & A \in \mathcal{S}_n \\
& & \vee\, (\exists p\, (n = 2p+1) \;\wedge\; \phi(n) = \dot{\forall} x\, B(x) \;\wedge\; A = (B(x_{p+1}) \mathrel{\dot{\Rightarrow}} \dot{\forall} x\, B(x)) \\
& & \vee\, (\exists p\, (n = 2p) \;\wedge\; (\mathcal{S}_n, A \vdash \bot \Rightarrow \mathcal{T}_0, \dot{\neg} A_0 \vdash \bot) \;\wedge\; A = \phi(n))
\end{array}
$$

Note however that $\mathcal{S}_n$ is used in negative position of an implication in the definition of $\mathcal{S}_{n+1}$. Hence, the complexity of the formula $A \in \mathcal{S}_n$ seen as a type of functions is a type of higher-order functions of depth n.

2.3. The computational content of the proof of completeness. We are now ready to formulate the proof as a program. We shall place ourselves in an axiom-free second-order intuitionistic arithmetic equipped with a proof-as-program interpretation[28], as already considered in Section 1.6. Additionally, we shall identify the construction of existentially quantified formulae and the construction of proofs of conjunctive formulae. For instance, we shall use the notation $(p_1, ..., p_n)$ for the proof of an n-ary combination of existential quantifiers and conjunctions. We shall also write $\mathit{dest}\ p\ \mathit{as}\ (x_1, ..., x_n)\ \mathit{in}\ q$ for a proof obtained by decomposition of the proof p of an n-ary combination of existential quantifiers and conjunctions. We shall write $\mathit{efq}\ p$ for a proof of A obtained by ex falso quodlibet from a proof p of $\bot$.

We shall use the letters n, A, Γ, m, p, q, r, h, g, f, k and their variants to refer to natural numbers, formulae, contexts of formulae, proofs of truth, proofs of derivability in the object language, proofs of belonging to $\mathcal{S}_\omega$, proofs of inconsistency from adding an implicative formula to $\mathcal{S}_{2n+1}$, proofs of belonging to $\mathcal{T}_0$, proofs of inclusion in $\mathcal{T}_0$, proofs of inclusion in extensions of $\mathcal{T}_0$, proofs of relative consistency, respectively.

The key property is $A \in \mathcal{S}_\omega$ which unfolds as $\exists n\, \exists \Gamma\, (\Gamma \subset \mathcal{S}_n \wedge \Gamma \vdash A)$. Rather than defining $\Gamma \subset \mathcal{S}_n$ from $A \in \mathcal{S}_n$ and the latter by cases, we now directly take $\Gamma \subset \mathcal{S}_n$ as our primitive concept, so that defining $A \in \mathcal{S}_n$ is actually not needed anymore. Rephrasing the property that $\mathcal{S}_n$ is inconsistent in terms of $\Gamma \subset \mathcal{S}_n$ is easy: it is enough to tell that $\Gamma \vdash \bot$ for some $\Gamma \subset \mathcal{S}_n$. In particular, we can write $\mathcal{S}_n, A \vdash \bot$ to mean $\exists \Gamma\, (\Gamma \subset \mathcal{S}_n \;\wedge\; \Gamma, A \vdash \bot)$.

We first define by cases the predicate $\Gamma \subset \mathcal{T}_0$:

$$
\frac{}{\epsilon \subset \mathcal{T}_0}\; J_{base} \qquad\qquad \frac{\Gamma \subset \mathcal{T}_0 \qquad A \in \mathcal{T}_0}{\Gamma, A \subset \mathcal{T}_0}\; J_{cons}
$$

Then, we can define $\Gamma \subset \mathcal{S}_n$ by cases: a formula B is allowed to occur in such Γ either because it is in $\mathcal{T}_0$ (clause J_{cons}), or because it is an Henkin axiom added at step $2n$ (clause $I_\forall$), or because it is an implication added at step $2n+1$ together with a proof of relative consistency of $\mathcal{S}_{n+1} \cup B$ with respect to $\mathcal{T}_0 \cup \dot{\neg} A_0$ (clause $I_\Rightarrow$). Note that we enforce in all cases that at least $\dot{\neg} A_0$ is in such Γ (clause I_0) and that we can always skip adding a formula at some step of the construction (clause I_S). Formally, the definition is:

[28]A typical effective framework for that purpose would be a fragment of the Calculus of Inductive Constructions such as it is implemented in the Coq proof assistant [16], or Matita [5]. The Calculus of Inductive Constructions is an impredicative extension of Martin-Löf's type theory [68].

$$\frac{\Gamma \subset \mathcal{T}_0}{\Gamma, \dot{\neg} A_0 \subset \mathcal{S}_0}\, I_0 \qquad \frac{\Gamma \subset \mathcal{S}_n}{\Gamma \subset \mathcal{S}_{n+1}}\, I_S \qquad \frac{\Gamma \subset \mathcal{S}_{2n}}{\Gamma,\ A(x_n) \dot{\Rightarrow} \dot{\forall} x\, A(x)\ \subset\ \mathcal{S}_{2n+1}}\, I_{\dot\forall}$$

$$\frac{\Gamma \subset \mathcal{S}_{2n+1} \qquad \mathcal{S}_{2n+1}, A \dot{\Rightarrow} B \vdash \bot \Rightarrow \mathcal{T}_0, \dot{\neg} A_0 \vdash \bot}{\Gamma,\ A \dot{\Rightarrow} B\ \subset\ \mathcal{S}_{2n+2}}\, I_{\dot\Rightarrow}$$

where $\phi(2n)$ is $\dot{\forall} x\, A(x)$ in $I_{\dot\forall}$ and $\phi(2n+1)$ is $A \dot{\Rightarrow} B$ in $I_{\dot\Rightarrow}$.

We now write as a program the proof that $\mathcal{S}_\omega$ is consistent relative to $\mathcal{T}_0 \cup \dot{\neg} A_0$, i.e. that $\bot \in \mathcal{S}_\omega$ implies $\mathcal{T}_0, \dot{\neg} A_0 \vdash \bot$. The latter expands to $\exists \Gamma (\Gamma \subset \mathcal{T}_0 \ \wedge\ \Gamma, \dot{\neg} A_0 \vdash \bot)$ which we see as made of triples of the form (Γ, g, p), with Γ a context, g a proof of $\Gamma \subset \mathcal{T}_0$ and p a proof of $\Gamma', \dot{\neg} A_0 \vdash \bot$.

This proof, which we call flush_n^Γ, takes as arguments a quadruple (n, Γ, f, p) where f is a proof of $\Gamma \subset \mathcal{S}_n$ and p proof of $\Gamma \vdash \bot$. It proceeds by cases on the proof of $\Gamma \subset \mathcal{S}_n$. When extended at odd n, it works by calling the *continuation* justifying that adding the formula $\phi(2p+1)$ preserves consistency, and, when extended at even n, by composing the resulting proof of inconsistency with a proof of the Drinker's paradox ($\mathit{dr\dot{i}nker}_y$ is the proof which builds a proof of $\Gamma \vdash \bot$ from a proof of $\Gamma, A(y) \dot{\Rightarrow} \dot{\forall} x\, A(x) \vdash \bot$, knowing that y does not occur in $\Gamma, \dot{\forall} x\, A(x)$, see Figure 2).

$$\begin{array}{llcl}
\mathit{flush} & : \quad \bot \in \mathcal{S}_\omega & \Rightarrow & \mathcal{T}_0, \dot{\neg} A_0 \vdash \bot \\
\mathit{flush} & (0, (\Gamma, \dot{\neg} A_0), I_0\, g, p) & \triangleq & (\Gamma, g, p) \\
\mathit{flush} & (n+1, \Gamma, I_S\, f, p) & \triangleq & \mathit{flush}\,(n, \Gamma, f, p) \\
\mathit{flush} & (2n+1, (\Gamma, A), \mathrm{I}_{\dot\forall}\, f, p) & \triangleq & \mathit{flush}\,(2n, \Gamma, f, \mathit{dr\dot{i}nker}_{x_n}\, p) \\
\mathit{flush} & (2n+2, (\Gamma, A), I_{\dot\Rightarrow}\,(f, k), p) & \triangleq & k\ (\Gamma, f, p)
\end{array}$$

A trivial lemma implicit in the natural language formulation of the proof of completeness is the lemma asserting $\dot{\neg} A_0 \subset \mathcal{S}_n$. The proof is by induction on n:

$$\begin{array}{lcl}
\mathit{inj}_n & : & \dot{\neg} A_0 \subset \mathcal{S}_n \\
\mathit{inj}_0 & \triangleq & I_0(J_{base}) \\
\mathit{inj}_{n+1} & \triangleq & I_S(\mathit{inj}_n)
\end{array}$$

A boring lemma which is implicit in the proof of completeness in natural language is that $\Gamma \subset \mathcal{S}_n$ and $\Gamma' \subset \mathcal{S}_{n'}$ imply $\Gamma \cup \Gamma' \subset \mathcal{S}_{max(n,n')}$. It looks obvious because one tends to think of $\Gamma \subset \mathcal{S}_n$ as denoting the inclusion of Γ within a uniquely defined relatively consistent set $\mathcal{S}_n$. However, the computational approach to the proof shows that $\mathcal{S}_n$ has no computational content per se: only proofs of $\Gamma \subset \mathcal{S}_n$ have, and such proofs are collections of proofs of relative consistency for only those implicative formulae which are in Γ. These formulae are those inspected by the lemma $A \in \mathcal{S}_\omega \Leftrightarrow \mathcal{M}_0 \vDash^e_\sigma A$, which in practice are subformulae of the formulae in $\mathcal{T}_0$.

For Γ_1 and Γ_2 included in $\mathcal{T}_0$, we write Γ_1, Γ_2 for their concatenation (possibly with redundancies). Otherwise, by construction, any Γ included in $\mathcal{S}_n$ for some n has either the form $\Gamma', \dot{\neg} A_0$ where Γ' is included in $\mathcal{T}_0$, or the form Γ', A where A has been added in the process of enumeration. We can then define $\Gamma_1 \cup \Gamma_2$ for Γ_1 and Γ_2 included in $\mathcal{S}_n$ for some n by cases[29]:

[29]Strictly speaking, this decomposition of any Γ included in $\mathcal{S}_n$ for some n should be part of the structure of Γ so as to be able to compute with it.

$$
\begin{array}{lcll}
\Gamma_1, \dot{\neg}A_0 & \cup & \Gamma_2, \dot{\neg}A_0 & ::= \ \Gamma_1, \Gamma_2, \dot{\neg}A_0 \\
\Gamma_1, \dot{\neg}A_0 & \cup & \Gamma_2, A & ::= \ (\Gamma_1, \dot{\neg}A_0 \ \cup \ \Gamma_2), A \\
\Gamma_1, A & \cup & \Gamma_2, \dot{\neg}A_0 & ::= \ (\Gamma_1 \ \cup \ \Gamma_2, \dot{\neg}A_0), A \\
\Gamma_1, A & \cup & \Gamma_2, A & ::= \ (\Gamma_1 \ \cup \ \Gamma_2), A \\
\Gamma_1, A & \cup & \Gamma_2, B & ::= \ (\Gamma_1, A \ \cup \ \Gamma_2), B \qquad \text{if } \lceil B \rceil > \lceil A \rceil \\
\Gamma_1, A & \cup & \Gamma_2, B & ::= \ (\Gamma_1 \ \cup \ \Gamma_2, B), A \qquad \text{if } \lceil A \rceil > \lceil B \rceil
\end{array}
$$

We can then define the merge of two proofs of $\Gamma \subset \mathcal{S}_n$ by distinguishing when the contexts are considered as subsets of $\mathcal{T}_0$ or as subsets of some $\mathcal{S}_n$:

$$
\begin{array}{llcllcl}
join_{\subset}^{\Gamma_1\Gamma_2} : & \Gamma_1 \subset \mathcal{T}_0 & \wedge & \Gamma_2 \subset \mathcal{T}_0 & & \Rightarrow & \Gamma_1, \Gamma_2 \subset \mathcal{T}_0 \\
join_{\subset}^{\Gamma_1\epsilon} & (\ g_1 & , & J_{base} &) & \triangleq & g_1 \\
join_{\subset}^{\Gamma_1(\Gamma_2,A_2)} & (\ g_1 & , & J_{cons}(g_2, h_2) &) & \triangleq & J_{cons}(join_{\subset}^{\Gamma_1\Gamma_2}(g_1, g_2), h_2) \\
 & & & & & & \\
join_{n}^{\Gamma_1\Gamma_2} : & \Gamma_1 \subset \mathcal{S}_n & \wedge & \Gamma_2 \subset \mathcal{S}_n & & \Rightarrow & \Gamma_1 \cup \Gamma_2 \subset \mathcal{S}_n \\
join_{0}^{(\Gamma_1,\dot{\neg}A_0)(\Gamma_2,\dot{\neg}A_0)} & (\ I_0(g_1) & , & I_0(g_2) &) & \triangleq & I_0(join_{\subset}^{\Gamma_1\Gamma_2}(g_1, g_2)) \\
join_{2n+1}^{(\Gamma_1A)(\Gamma_2A)} & (\ \mathtt{I}_\forall(f_1) & , & \mathtt{I}_\forall(f_2) &) & \triangleq & \mathtt{I}_\forall(join_{2n}^{\Gamma_1\Gamma_2}(f_1, f_2)) \\
join_{2n+2}^{(\Gamma_1A)(\Gamma_2A)} & (\ \mathtt{I}_\Rightarrow(f_1, k_1) & , & \mathtt{I}_\Rightarrow(f_2, k_2) &) & \triangleq & \mathtt{I}_\Rightarrow(join_{2n+1}^{\Gamma_1\Gamma_2}(f_1, f_2), k_1) \\
join_{2n+1}^{(\Gamma_1A)\Gamma_2} & (\ \mathtt{I}_\forall(f_1) & , & I_S(f_2) &) & \triangleq & \mathtt{I}_\forall(join_{2n}^{\Gamma_1\Gamma_2}(f_1, f_2)) \\
join_{2n+2}^{(\Gamma_1A)\Gamma_2} & (\ \mathtt{I}_\Rightarrow(f_1, k_1) & , & I_S(f_2) &) & \triangleq & \mathtt{I}_\Rightarrow(join_{2n+1}^{\Gamma_1\Gamma_2}(f_1, f_2), k_1) \\
join_{2n+1}^{\Gamma_1(\Gamma_2A)} & (\ I_S(f_1) & , & \mathtt{I}_\forall(f_2) &) & \triangleq & \mathtt{I}_\forall(join_{2n}^{\Gamma_1\Gamma_2}(f_1, f_2)) \\
join_{2n+2}^{\Gamma_1(\Gamma_2A)} & (\ I_S(f_1) & , & \mathtt{I}_\Rightarrow(f_2, k_2) &) & \triangleq & \mathtt{I}_\Rightarrow(join_{2n+1}^{\Gamma_1\Gamma_2}(f_1, f_2), k_2) \\
join_{n+1}^{\Gamma_1\Gamma_2} & (\ I_S(f_1) & , & I_S(f_2) &) & \triangleq & I_S(join_{n}^{\Gamma_1\Gamma_2}(f_1, f_2)) \\
 & & & & & & \\
hjoin_{n_1n_2}^{\Gamma_1\Gamma_2} : & \Gamma_1 \subset \mathcal{S}_{n_1} & \wedge & \Gamma_2 \subset \mathcal{S}_{n_2} & & \Rightarrow & \Gamma_1 \cup \Gamma_2 \subset \mathcal{S}_{max(n_1,n_2)} \\
hjoin_{nn}^{\Gamma_1\Gamma_2} & (\ f_1 & , & f_2 &) & \triangleq & join_{n}^{\Gamma_1\Gamma_2}(f_1, f_2) \\
hjoin_{n_1'+1>n_2}^{\Gamma_1\Gamma_2} & (\ I_S(f_1) & , & f_2 &) & \triangleq & I_S(hjoin_{n_1'n_2}^{\Gamma_1\Gamma_2}(f_1, f_2)) \\
hjoin_{n_1'+1>n_2}^{(\Gamma_1A_1)\Gamma_2} & (\ I_\forall(f_1) & , & f_2 &) & \triangleq & I_\forall(hjoin_{n_1'n_2}^{\Gamma_1\Gamma_2}(f_1, f_2)) \\
hjoin_{n_1'+1>n_2}^{(\Gamma_1A_1)\Gamma_2} & (\ I_\Rightarrow(f_1, k_1) & , & f_2 &) & \triangleq & I_\Rightarrow(hjoin_{n_1'n_2}^{\Gamma_1\Gamma_2}(f_1, f_2), k_1) \\
hjoin_{n_1<n_2'+1}^{\Gamma_1\Gamma_2} & (\ f_1 & , & I_S(f_2) &) & \triangleq & I_S(hjoin_{n_1n_2'}^{\Gamma_1\Gamma_2}(f_1, f_2)) \\
hjoin_{n_1<n_2'+1}^{\Gamma_1(\Gamma_2A_2)} & (\ f_1 & , & \mathtt{I}_\forall(f_2) &) & \triangleq & \mathtt{I}_\forall(hjoin_{n_1n_2'}^{\Gamma_1\Gamma_2}(f_1, f_2)) \\
hjoin_{n_1<n_2'+1}^{\Gamma_1(\Gamma_2A_2)} & (\ f_1 & , & \mathtt{I}_\Rightarrow(f_2, k_2) &) & \triangleq & \mathtt{I}_\Rightarrow(hjoin_{n_1n_2'}^{\Gamma_1\Gamma_2}(f_1, f_2), k_2)
\end{array}
$$

In particular, it has to be noticed that the merge possibly does arbitrary choices: when the same implicative formula A occurs in both contexts, only one of the two proofs telling how to reduce $\Gamma, A \vdash \bot$ to $\mathcal{T}_0, \dot{\neg}A_0 \vdash \bot$ (third clause of $join_n$) is (arbitrarily) kept.

Another combinatoric lemma is that the merge of contexts indeed produces a bigger context. To state the lemma, we already need to define the inclusion of contexts $\Gamma \subset \Gamma'$. This can be done inductively by the following clauses:

$$
\frac{}{\epsilon \subset \epsilon}\ L_0^{\subset} \qquad \frac{\Gamma \subset \Gamma'}{\Gamma \subset \Gamma', A}\ L_N^{\subset} \qquad \frac{\Gamma \subset \Gamma'}{\Gamma, A \subset \Gamma', A}\ L_S^{\subset}
$$

Two straightforward lemmas are that $\Gamma_1 \subset \Gamma_1, \Gamma_2$ and $\Gamma_2 \subset \Gamma_1, \Gamma_2$ for Γ_1 and Γ_2 included in $\mathcal{T}_0$. The proofs are by induction on Γ_1 and Γ_2 where i is either 1 or 2:

$$
\begin{array}{lcl}
incl_i^{\Gamma_1,\Gamma_2} & : & \Gamma_i \subset \Gamma_1, \Gamma_2 \\
incl_i^{\epsilon,\epsilon} & \triangleq & L_0 \\
incl_1^{(\Gamma_1,A),\epsilon} & \triangleq & L_S(incl_1^{\Gamma_1\epsilon}) \\
incl_2^{(\Gamma_1,A),\epsilon} & \triangleq & L_N(incl_2^{\Gamma_1\epsilon}) \\
incl_1^{\Gamma_1,(\Gamma_2,A)} & \triangleq & L_N(incl_1^{\Gamma_1\Gamma_2}) \\
incl_2^{\Gamma_1,(\Gamma_2,A)} & \triangleq & L_S(incl_2^{\Gamma_1\Gamma_2})
\end{array}
$$

This allows to prove the following lemma where i is either 1 or 2:

$$
\begin{array}{lcll}
incl'^{\Gamma_1,\Gamma_2}_i & : & \Gamma_i \subset \Gamma_1 \cup \Gamma_2 & \\
incl'^{(\Gamma_1,\dot{\neg}A_0),(\Gamma_2,\dot{\neg}A_0)}_i & \triangleq & L_S(incl_i^{\Gamma_1,\Gamma_2}) & \\
incl'^{(\Gamma_1,\dot{\neg}A_0),(\Gamma_2,A)}_1 & \triangleq & L_N(incl'^{(\Gamma_1,\dot{\neg}A_0),\Gamma_2}_1) & \\
incl'^{(\Gamma_1,\dot{\neg}A_0),(\Gamma_2,A)}_2 & \triangleq & L_S(incl'^{(\Gamma_1,\dot{\neg}A_0),\Gamma_2}_2) & \\
incl'^{(\Gamma_1,A),(\Gamma_2,\dot{\neg}A_0)}_1 & \triangleq & L_S(incl'^{\Gamma_1,(\Gamma_2,\dot{\neg}A_0)}_1) & \\
incl'^{(\Gamma_1,A),(\Gamma_2,\dot{\neg}A_0)}_2 & \triangleq & L_N(incl'^{\Gamma_1,(\Gamma_2,\dot{\neg}A_0)}_2) & \\
incl'^{(\Gamma_1,A),(\Gamma_2,A)}_i & \triangleq & L_S(incl'^{\Gamma_1,\Gamma_2}_i) & \\
incl'^{(\Gamma_1,A),(\Gamma_2,B)}_1 & \triangleq & L_N(incl'^{(\Gamma_1,A),\Gamma_2}_1) & \text{if } \lceil A \rceil < \lceil B \rceil \\
incl'^{(\Gamma_1,A),(\Gamma_2,B)}_2 & \triangleq & L_S(incl'^{(\Gamma_1,A),\Gamma_2}_2) & \text{if } \lceil A \rceil < \lceil B \rceil \\
incl'^{(\Gamma_1,A),(\Gamma_2,B)}_1 & \triangleq & L_S(incl'^{\Gamma_1,(\Gamma_2,B)}_1) & \text{if } \lceil A \rceil > \lceil B \rceil \\
incl'^{(\Gamma_1,A),(\Gamma_2,B)}_2 & \triangleq & L_N(incl'^{\Gamma_1,(\Gamma_2,B)}_2) & \text{if } \lceil A \rceil > \lceil B \rceil
\end{array}
$$

Our object logic is defined by the rules on Figure 2. Note that we shall use non standard derived rules. For instance, we shall not use the rule $\dot{abs}^{\Rightarrow}$ and $\dot{abs}^{\forall}$ but instead the derived rules $\pi_1^{\Rightarrow}, \pi_2^{\Rightarrow}$ and $dr\dot{i}nker_y$.

Thanks to the previous lemma, we are able to translate proofs of $A_1 \in \mathcal{S}_\omega$ and $A_2 \in \mathcal{S}_\omega$ living in possibly two different contexts to eventually live in the union of the two contexts:

$$
\begin{array}{llll}
share : A_1 \in \mathcal{S}_\omega & \wedge\ A_2 \in \mathcal{S}_\omega & \Rightarrow & \exists n \exists \Gamma (\Gamma \subset \mathcal{S}_n \wedge \Gamma \vdash A_1 \wedge \Gamma \vdash A_2) \\
share \quad (n_1, \Gamma_1, f_1, p_1) & (n_2, \Gamma_2, f_2, p_2) & \triangleq & \left(\begin{array}{l} max(n_1, n_2), (\Gamma_1 \cup \Gamma_2), hjoin(f_1, f_2), \\ w\dot{e}ak(incl'^{\Gamma_1,\Gamma_2}_1, p_1), \\ w\dot{e}ak(incl'^{\Gamma_1,\Gamma_2}_2, p_2) \end{array} \right)
\end{array}
$$

where $w\dot{e}ak$ is an admissible rule of the object logic.

Thanks to the ability to ensure distinct proofs to live in the same context, we can reformulate the relevant rules of the object logic as rules over $\mathcal{S}_\omega$, as well as provide proofs of specific formulae:

$$\begin{array}{llllll}
\dot{APP}^{\Rightarrow} : & A \mathrel{\dot{\Rightarrow}} B \in \mathcal{S}_\omega & \wedge\ A \in \mathcal{S}_\omega & \Rightarrow & B \in \mathcal{S}_\omega \\
\dot{APP}^{\Rightarrow} & q & q' & \triangleq & \begin{array}{l} \mathit{dest\ share}\,(q,q')\,\mathit{as}\,(n',\Gamma',f',p'p'')\,\mathit{in} \\ (n',\Gamma',f',\dot{app}^{\Rightarrow}(p',p'')) \end{array}
\end{array}$$

$$\begin{array}{llllll}
\dot{APP}^{\dot\forall} : & \dot\forall x\,A(x) \in \mathcal{S}_\omega & \wedge\ \mathcal{T}erm & \Rightarrow & A(t) \in \mathcal{S}_\omega \\
\dot{APP}^{\dot\forall} & (n,\Gamma,f,p) & t & \triangleq & (n,\Gamma,f,\dot{app}^{\dot\forall}(p,t))
\end{array}$$

$$\begin{array}{llll}
\dot{DN} : & \dot\neg\dot\neg A \in \mathcal{S}_\omega & \Rightarrow & A \in \mathcal{S}_\omega \\
\dot{DN} & (n,\Gamma,f,p) & \triangleq & (n,\Gamma,f,\dot{dn}(p))
\end{array}$$

$$\begin{array}{lll}
\dot{AX}_{\dot\forall x A(x)} & : & A(x_n) \mathrel{\dot\Rightarrow} \dot\forall x\,A(x) \in \mathcal{S}_\omega \\
\dot{AX}_{\dot\forall x A(x)} & \triangleq & (2n+1, (\dot\neg A_0, A(x_n) \mathrel{\dot\Rightarrow} \dot\forall x\,A(x)), \mathtt{I}_\forall(inj_{2n}), \dot{ax}_0) \\
 & & \hfill \mathit{where}\ 2n = \ulcorner \dot\forall x\,A(x) \urcorner
\end{array}$$

$$\begin{array}{lll}
\dot{AX}^0_{\dot\neg A_0} & : & \dot\neg A_0 \in \mathcal{S}_\omega \\
\dot{AX}^0_{\dot\neg A_0} & \triangleq & (0, \dot\neg A_0, J_{base}, \dot{ax}_0)
\end{array}$$

Similarly, we can formulate rules on provability in $\mathcal{T}_0$:

$$\begin{array}{llll}
\dot{DNABS} : & \mathcal{T}_0, \dot\neg A_0 \vdash \dot\bot & \Rightarrow & \mathcal{T}_0 \vdash A_0 \\
\dot{DNABS} & (\Gamma, g, p) & \triangleq & (\Gamma, g, \dot{dn}(\dot{abs}^{\Rightarrow}(p)))
\end{array}$$

$$\begin{array}{llll}
\dot{BOT} : & \mathcal{T}_0, \dot\neg A_0 \vdash \dot\bot & \Rightarrow & \dot\bot \in \mathcal{S}_\omega \\
\dot{BOT} & (\Gamma, g, p) & \triangleq & (0, (\Gamma, \dot\neg A_0), I_0(g), p)
\end{array}$$

When $\ulcorner A \mathrel{\dot\Rightarrow} B \urcorner = 2n+1$, we can also derive the following properties:

$$\begin{array}{llll}
\dot{AX}_{A \dot\Rightarrow B} : & ((\mathcal{S}_{2n+1}, A \mathrel{\dot\Rightarrow} B \vdash \dot\bot) \Rightarrow (\mathcal{T}_0, \dot\neg A_0 \vdash \dot\bot)) & \Rightarrow & A \mathrel{\dot\Rightarrow} B \in \mathcal{S}_\omega \\
\dot{AX}_{A \dot\Rightarrow B} & k & \triangleq & \left(\begin{array}{l} 2n+2, (\dot\neg A_0, A \mathrel{\dot\Rightarrow} B), \\ I_\Rightarrow(inj_{2n+1}, k), \dot{ax}_0 \end{array} \right)
\end{array}$$

$$\begin{array}{llll}
\dot{PROJ}_1^{\Rightarrow} : & \mathcal{S}_{2n+1}, A \mathrel{\dot\Rightarrow} B \vdash \dot\bot & \Rightarrow & A \in \mathcal{S}_\omega \\
\dot{PROJ}_1^{\Rightarrow} & (\Gamma, f, p) & \triangleq & (2n+1, \Gamma, f, \pi_1^{\Rightarrow}\ p)
\end{array}$$

$$\begin{array}{llll}
\dot{PROJ}_2^{\Rightarrow} : & \mathcal{S}_{2n+1}, A \mathrel{\dot\Rightarrow} B \vdash \dot\bot & \Rightarrow & \dot\neg B \in \mathcal{S}_\omega \\
\dot{PROJ}_2^{\Rightarrow} & (\Gamma, f, p) & \triangleq & (2n+1, \Gamma, f, \pi_2^{\Rightarrow}\ p)
\end{array}$$

We are now ready to present the main computational piece of the completeness proof and we shall use for that notations reminiscent from semantic normalisation [19], or type-directed partial evaluation [23], as considered when proving completeness of intuitionistic logic with respect to models such a Kripke or Beth models.

We have to prove $\mathcal{M}_0 \vDash^e_\sigma A \Leftrightarrow A[\sigma] \in \mathcal{S}_\omega$, which means proving $\mathcal{M}_0 \vDash^e_\sigma A \Rightarrow A[\sigma] \in \mathcal{S}_\omega$ and $A[\sigma] \in \mathcal{S}_\omega \Rightarrow \mathcal{M}_0 \vDash^e_\sigma A$. As in semantic normalisation (see Section 1.6), we shall call *reification* and write $\downarrow^A_\sigma$ the proof mapping a semantic formula (i.e. $\mathcal{M}_0 \vDash^e_\sigma A$) to a syntactic formula, i.e. $A[\sigma] \in \mathcal{S}_\omega$. We shall call *reflection* and write $\uparrow^A_\sigma$ for the way up going from the syntactic view to the semantic view.

Primitive rules

$$\frac{|\Gamma'| = i}{\Gamma, A, \Gamma' \vdash A}\ \dot{ax}_i \qquad \frac{\Gamma \vdash \dot{\neg}\dot{\neg}A}{\Gamma \vdash A}\ \dot{dn}$$

$$\frac{\Gamma \vdash A \dot{\Rightarrow} B \qquad \Gamma' \vdash A}{\Gamma \cup \Gamma' \vdash B}\ \dot{app}^{\dot{\Rightarrow}} \qquad \frac{\Gamma \vdash \dot{\forall} x\, A(x)}{\Gamma \vdash A(t)}\ \dot{app}^{\dot{\forall}}{}_t$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \dot{\Rightarrow} B}\ \dot{abs}^{\dot{\Rightarrow}} \qquad \frac{\Gamma \vdash A(y) \qquad y \text{ not in } \dot{\forall} x\, A(x), \Gamma}{\Gamma \vdash \dot{\forall} x\, A(x)}\ \dot{abs}^{\dot{\forall}}$$

Admissible rules

$$\frac{\Gamma, A(y) \dot{\Rightarrow} \dot{\forall} x\, A(x) \vdash \dot{\bot} \qquad y \text{ not in } \dot{\forall} x\, A(x), \Gamma}{\Gamma \vdash \dot{\bot}}\ \dot{drinker}_y$$

$$\frac{\Gamma, A \dot{\Rightarrow} B \vdash \dot{\bot}}{\Gamma \vdash A}\ \pi_1^{\dot{\Rightarrow}} \qquad \frac{\Gamma, A \dot{\Rightarrow} B \vdash \dot{\bot}}{\Gamma \vdash \dot{\neg} B}\ \pi_2^{\dot{\Rightarrow}} \qquad \frac{\Gamma \vdash \dot{\bot}}{\Gamma \vdash A}\ efq \qquad \frac{\Gamma \subset \Gamma' \qquad \Gamma \vdash A}{\Gamma' \vdash A}\ \dot{weak}$$

FIGURE 2. Inference rules characterising classical first-order predicate calculus

$$\begin{array}{llcl}
\downarrow^A_\sigma : & \mathcal{M}_0 \vDash^e_\sigma A & \Rightarrow & A[\sigma] \in \mathcal{S}_\omega \\
\downarrow^{P(\vec{t})}_\sigma & m & \triangleq & m \\
\downarrow^{\dot{\bot}}_\sigma & m & \triangleq & \dot{BOT}(m) \\
\downarrow^{A \dot{\Rightarrow} B}_\sigma & m & \triangleq & \dot{AX}_{A[\sigma] \dot{\Rightarrow} B[\sigma]}(kont^{A \dot{\Rightarrow} B}_\sigma(m)) \\
\downarrow^{\dot{\forall} x A}_\sigma & m & \triangleq & \dot{APP}^{\dot{\Rightarrow}}(\dot{AX}_{(\dot{\forall} x A)[\sigma]}, \downarrow^A_{\sigma, x \leftarrow x_n}(m\, x_n)) \text{ where } 2n = \ulcorner(\dot{\forall} x\, A)[\sigma]\urcorner \\
 & & & \\
\uparrow^A_\sigma : & A[\sigma] \in \mathcal{S}_\omega & \Rightarrow & \mathcal{M}_0 \vDash^e_\sigma A \\
\uparrow^{P(\vec{t})}_\sigma & q & \triangleq & q \\
\uparrow^{\dot{\bot}}_\sigma & q & \triangleq & \mathit{flush}\, q \\
\uparrow^{A \dot{\Rightarrow} B}_\sigma & q & \triangleq & m \mapsto \uparrow^B_\sigma (\dot{APP}^{\dot{\Rightarrow}}(q, \downarrow^A_\sigma m)) \\
\uparrow^{\dot{\forall} x A}_\sigma & q & \triangleq & t \mapsto \uparrow^A_{\sigma, x \leftarrow t}(\dot{APP}^{\dot{\forall}}(q, t))
\end{array}$$

where, for m proving $\mathcal{M}_0 \vDash^e_\sigma A \dot{\Rightarrow} B$, the relative consistency proof $kont^{A \dot{\Rightarrow} B}_\sigma(m)$ is defined by:

$$\begin{array}{l}
kont^{A \dot{\Rightarrow} B}_\sigma(m) \ : \ (\mathcal{S}_{\ulcorner A[\sigma] \dot{\Rightarrow} B[\sigma]\urcorner}, A[\sigma] \dot{\Rightarrow} B[\sigma] \vdash \dot{\bot}) \Rightarrow (\mathcal{T}_0, \dot{\neg} A_0 \vdash \dot{\bot}) \\
kont^{A \dot{\Rightarrow} B}_\sigma(m) \triangleq r \mapsto \mathit{flush}\, (\dot{APP}^{\dot{\Rightarrow}}(\dot{PROJ}_2^{\dot{\Rightarrow}}(r), \downarrow^B_\sigma (m\, (\uparrow^A_\sigma (\dot{PROJ}_1^{\dot{\Rightarrow}}(r))))))
\end{array}$$

We still have to prove that the model is classical, which we do by lifting the double-negation elimination rule to the semantics:

$$\begin{array}{lcl}
classic_0 & : & \forall A\, ((\mathcal{M}_0 \vDash^e_{id} \dot{\neg}\dot{\neg} A) \Rightarrow (\mathcal{M}_0 \vDash^e_{id} A)) \\
classic_0 & \triangleq & A \mapsto m \mapsto \uparrow^A_{id} (\dot{DN}(\downarrow^{\dot{\neg}\dot{\neg} A}_{id} m))
\end{array}$$

It remains also to show that every formula of $\mathcal{T}_0$ is true in $\mathcal{M}_0$ and this is obtained by the axiom rule:

$$\begin{array}{lcl} init_0 & : & \forall B \in \mathcal{T}_0\, \mathcal{M}_0 \models^e_{id} B \\ init_0 & \triangleq & B \mapsto h \mapsto \uparrow^B_{id} (0, (B, \dot{\neg} A_0), I_0(J_{cons}(J_{base}, h)), \dot{ax}_1) \end{array}$$

Finally, we get the completeness result stated as S1' by:

$$\begin{array}{lcl} completeness & : & \forall \mathcal{M} \forall \sigma\, (\mathcal{M} \models^e_\sigma Classic \Rightarrow \mathcal{M} \models^e_\sigma \mathcal{T}_0 \Rightarrow \mathcal{M} \models^e_\sigma A_0) \Rightarrow \mathcal{T}_0 \vdash A_0 \\ completeness & \triangleq & \psi \mapsto D\dot{N}ABS\, (flush\, (AP\dot{P}^{\Rightarrow} (A\dot{X}^0_{\dot{\neg} A_0}, \downarrow^{A_0}_{id} (\psi\, \mathcal{M}_0\; id\; classic_0\; init_0)))) \end{array}$$

Notice that the final result is a triple (Γ, g, p) such that p is a proof of $\Gamma \vdash A_0$ and g is a proof of $\Gamma \subset \mathcal{T}_0$.

2.4. The computational content on examples. To illustrate the behaviour of the completeness proofs, we consider two examples. We use notations of λ-calculus to represent proofs in the meta-logic and constructors from Figure 2 for proofs in the object logic.

We place ourselves in the empty theory and consider the formula $A_0 \triangleq X \dot{\Rightarrow} Y \dot{\Rightarrow} X$ with X and Y propositional atoms.

The expansion of $\models^e A_0$ is $\forall \mathcal{M}\ \forall \sigma\ (\sigma \models^e_{\mathcal{M}} Classic \Rightarrow \mathcal{M} \models^e X \Rightarrow \mathcal{M} \models^e Y \Rightarrow \mathcal{M} \models^e X)$. It has a canonical proof, which, as a λ-term, is the closure of the so-called combinator K over the symbols it depends on:

$$m \triangleq (\mathcal{D}, \mathcal{F}, \mathcal{P}, B) \mapsto \sigma \mapsto c \mapsto (x : \mathcal{P}(X)) \mapsto (y : \mathcal{P}(Y)) \mapsto x$$

Applying completeness means instantiating the model by the syntactic model and the substitution by the empty substitution so as to obtain from m the proof

$$m_0 \triangleq (x : X \in \mathcal{S}_\omega) \mapsto (y : Y \in \mathcal{S}_\omega) \mapsto x$$

Our object proof is then the result of evaluating

$$D\dot{N}ABS\, (flush\, (AP\dot{P}^{\Rightarrow} (A\dot{X}^0_{\dot{\neg} A_0}, \downarrow^{A_0} m_0)))$$

which proceeds as follows:

$$D\dot{N}ABS\, (flush\, (AP\dot{P}^{\Rightarrow} (A\dot{X}^0_{\dot{\neg} A_0}, A\dot{X}_{A_0}(kont^{A_0}(m_0)))))$$

where $kont^{A_0}(m_0)(r)$ reduces to $flush\, (AP\dot{P}^{\Rightarrow} (PR\dot{O}J_2^{\dot{\Rightarrow}} (r), \downarrow^{Y \dot{\Rightarrow} X} (m_0\, (\uparrow^X (PR\dot{O}J_1^{\dot{\Rightarrow}} (r))))))$ for r proving $\mathcal{S}_{\lceil A_0 \rceil}, A_0 \vdash \bot$, that is to

$$flush\, (AP\dot{P}^{\Rightarrow} (PR\dot{O}J_2^{\dot{\Rightarrow}} (r), A\dot{X}_{Y \dot{\Rightarrow} X}(kont^{Y \dot{\Rightarrow} X}(m_0\, (PR\dot{O}J_1^{\dot{\Rightarrow}} (r))))))$$

In there, $kont^{Y \dot{\Rightarrow} X}(m_0\, (PR\dot{O}J_1^{\dot{\Rightarrow}} (r)))(r')$, for r' proving $\mathcal{S}_{\lceil Y \dot{\Rightarrow} X \rceil}, Y \dot{\Rightarrow} X \vdash \bot$, reduces in turn to

$$flush\, (AP\dot{P}^{\Rightarrow} (PR\dot{O}J_2^{\dot{\Rightarrow}} (r'), m_0\, (PR\dot{O}J_1^{\dot{\Rightarrow}} (r))\, (PR\dot{O}J_1^{\dot{\Rightarrow}} (r'))))$$

Evaluating $AP\dot{P}^{\Rightarrow} (A\dot{X}^0_{\dot{\neg} A_0}, A\dot{X}_{A_0}(kont^{A_0}(m_0)))$ gives a tuple

$$(\lceil A_0 \rceil, (\dot{\neg} A_0, A_0), I_{\Rightarrow}(inj_{\lceil A_0 \rceil}, kont^{A_0}(m_0)), p_0)$$

where $p_0 \triangleq \dot{app}^{\Rightarrow} (\dot{ax}_1, \dot{ax}_0)$ is a proof of $\dot{\neg} A_0, A_0 \vdash \bot$ obtained by application of the two axiom rules proving $\dot{\neg} A_0 \vdash \dot{\neg} A_0$ and $\dot{\neg} A_0, A_0 \vdash A_0$.

Evaluating the outermost *flush* triggers the application of the continuation $kont^{A_0}(m_0)$ to $r_0 \triangleq (\dot{\neg} A_0, inj_{\lceil A_0 \rceil}, p_0)$, meaning that the whole object proof becomes

$$D\dot{N}ABS\, (flush\, (AP\dot{P}^{\Rightarrow} (PR\dot{O}J_2^{\dot{\Rightarrow}} (r_0), A\dot{X}_{Y \dot{\Rightarrow} X}(kont^{Y \dot{\Rightarrow} X}(m_0\, (PR\dot{O}J_1^{\dot{\Rightarrow}} (r_0)))))))$$

Evaluating $A\dot{P}P^{\dot\Rightarrow}(PR\dot{O}J_2^{\dot\Rightarrow}(r_0), A\dot{X}_{Y\dot\Rightarrow X}(kont^{Y\dot\Rightarrow X}(m_0\,(PR\dot{O}J_1^{\dot\Rightarrow}(r_0)))))$ gives a tuple

$$(\lceil Y \dot\Rightarrow X\rceil, (\dot\neg A_0, Y \dot\Rightarrow X), I_{\Rightarrow}(inj_{\lceil Y\dot\Rightarrow X\rceil}, kont^{Y\dot\Rightarrow X}(m_0\,(PR\dot{O}J_1^{\dot\Rightarrow}(r_0)))), p_1)$$

where $p_1 \triangleq a\dot{p}p^{\dot\Rightarrow}(\pi_2^{\dot\Rightarrow}(p_0), \dot{a}x_0)$ is a proof of $\dot\neg A_0, Y \dot\Rightarrow X \vdash \dot\bot$.

Evaluating the new outermost *flush* triggers in turn the application of the continuation $kont^{Y\dot\Rightarrow X}(m_0\,(PR\dot{O}J_1^{\dot\Rightarrow}(r_0)))$ to $r_1 \triangleq (\dot\neg A_0, inj_{\lceil Y\dot\Rightarrow X\rceil}, p_1)$ and this results in

(1) $$DN\dot{A}BS\,(flush\,(A\dot{P}P^{\dot\Rightarrow}(PR\dot{O}J_2^{\dot\Rightarrow}(r_1), m_0\,(PR\dot{O}J_1^{\dot\Rightarrow}(r_0))\,(PR\dot{O}J_1^{\dot\Rightarrow}(r_1)))))$$

that is, taking into account the definition of m_0

$$DN\dot{A}BS\,(flush\,(A\dot{P}P^{\dot\Rightarrow}(PR\dot{O}J_2^{\dot\Rightarrow}(r_1), PR\dot{O}J_1^{\dot\Rightarrow}(r_0))))$$

No continuations are produced by $A\dot{P}P^{\dot\Rightarrow}(PR\dot{O}J_2^{\dot\Rightarrow}(r_1), PR\dot{O}J_1^{\dot\Rightarrow}(r_0))$ so the only role of the last *flush* is to peel the I_S leading to a proof $r_2 \triangleq (\epsilon, J_{base}, \dot{d}n(p_2))$ where $p_2 \triangleq a\dot{p}p^{\dot\Rightarrow}(\pi_2^{\dot\Rightarrow}(p_1), \pi_1^{\dot\Rightarrow}(p_0))$ combines a proof of $\dot\neg A_0 \vdash \dot\neg X$ with a proof of $\dot\neg A_0 \vdash X$ to get a proof of $\dot\neg A_0 \vdash \dot\bot$.

To summarise, the object proof produced is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots\; p_0}{\dot\neg A_0, A_0 \vdash \dot\bot}}{\dot\neg A_0 \vdash \dot\neg(Y\dot\Rightarrow X)}\;\pi_2^{\dot\Rightarrow} \qquad \dfrac{}{\dot\neg A_0, Y\dot\Rightarrow X \vdash Y \dot\Rightarrow X}\;\dot{a}x}{\dot\neg A_0, Y\dot\Rightarrow X \vdash \dot\bot}\;a\dot{p}p^{\dot\Rightarrow}}{\dot\neg A_0 \vdash \dot\neg X}\;\pi_2^{\dot\Rightarrow} \qquad \dfrac{\dfrac{\vdots\; p_0}{\dot\neg A_0, A_0 \vdash \dot\bot}}{\dot\neg A_0 \vdash X}\;\pi_1^{\dot\Rightarrow}}{\dfrac{\dot\neg A_0 \vdash \dot\bot}{\dfrac{\vdash \dot\neg\dot\neg A_0}{\vdash A_0}\;\dot{d}n}\;a\dot{b}s^{\dot\Rightarrow}}\;a\dot{p}p^{\dot\Rightarrow}$$

where p_0 is:

$$\dfrac{\dfrac{}{\dot\neg A_0 \vdash \dot\neg A_0}\;\dot{a}x \qquad \dfrac{}{A_0 \vdash A_0}\;\dot{a}x}{\dot\neg A_0, A_0 \vdash \dot\bot}\;a\dot{p}p^{\dot\Rightarrow}$$

As a matter of comparison, for the canonical proof of the validity of $A_0' \triangleq X \dot\Rightarrow Y \dot\Rightarrow Y$, everything up to step (1) above is the same modulo the change of A_0 into A_0' and of $Y \dot\Rightarrow X$ into $Y \dot\Rightarrow Y$. After step (1), one obtains

$$DN\dot{A}BS\,(flush\,(A\dot{P}P^{\dot\Rightarrow}(PR\dot{O}J_2^{\dot\Rightarrow}(r_1'), PR\dot{O}J_1^{\dot\Rightarrow}(r_1'))))$$

where

$$\begin{array}{rcl} r_1' & \triangleq & (\dot\neg A_0', inj_{\lceil Y\dot\Rightarrow Y\rceil}, p_1') \\ p_1' & \triangleq & a\dot{p}p^{\dot\Rightarrow}(\pi_2^{\dot\Rightarrow}(p_0'), \dot{a}x_0) \\ p_0' & \triangleq & a\dot{p}p^{\dot\Rightarrow}(\dot{a}x_1, \dot{a}x_0) \end{array}$$

Finally, this yields $(\epsilon, J_{base}, \dot{dn}(p'_2))$ where $p'_2 \triangleq \dot{app}^{\dot{\Rightarrow}}(\pi_2^{\dot{\Rightarrow}}(p'_1), \pi_1^{\dot{\Rightarrow}}(p'_1))$, that is, graphically:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots\; p'_1}{\dot{\neg}A'_0, Y \dot{\Rightarrow} Y \vdash \dot{\bot}}}{\dot{\neg}A'_0 \vdash \dot{\neg}Y}\,\pi_2^{\dot{\Rightarrow}} \qquad \dfrac{\dfrac{\vdots\; p'_1}{\dot{\neg}A'_0, Y \dot{\Rightarrow} Y \vdash \dot{\bot}}}{\dot{\neg}A'_0 \vdash Y}\,\pi_1^{\dot{\Rightarrow}}}{\dot{\neg}A'_0 \vdash \dot{\bot}}\,\dot{app}^{\dot{\Rightarrow}}}{\vdash \dot{\neg}\dot{\neg}A'_0}\,\dot{abs}^{\dot{\Rightarrow}}}{\vdash A'_0}\,\dot{dn}
$$

where, graphically, p'_1 is:

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\dot{\neg}A'_0 \vdash \dot{\neg}A'_0}\,\dot{ax} \qquad \dfrac{}{A'_0 \vdash A'_0}\,\dot{ax}}{\dot{\neg}A'_0, A'_0 \vdash \dot{\bot}}\,\dot{app}^{\dot{\Rightarrow}}}{\dot{\neg}A'_0 \vdash \dot{\neg}(Y \dot{\Rightarrow} Y)}\,\pi_2^{\dot{\Rightarrow}} \qquad \dfrac{}{\dot{\neg}A'_0, Y \dot{\Rightarrow} Y \vdash Y \dot{\Rightarrow} Y}\,\dot{ax}}{\dot{\neg}A'_0, Y \dot{\Rightarrow} Y \vdash \dot{\bot}}
$$

We can notice in particular that, treating the metalanguage as a λ-calculus as we did, the two canonical proofs of validity of $X \dot{\Rightarrow} X \dot{\Rightarrow} X$ would not produce the same object language proofs.

2.5. Extension to conjunction. Henkin's original proof [39] includes only implication, universal quantification and the false connective. Handling conjunction in our presentation of Henkin's proof is straightforward. Let us assume the object language being equipped with the following rules for conjunction:

$$
\dfrac{\Gamma \vdash A_1 \quad \Gamma \vdash A_2}{\Gamma \vdash A_1 \dot{\wedge} A_2}\,\dot{pair} \qquad \dfrac{\Gamma \vdash A_1 \dot{\wedge} A_2}{\Gamma \vdash A_1}\,\dot{\pi}_1^{\wedge} \qquad \dfrac{\Gamma \vdash A_1 \dot{\wedge} A_2}{\Gamma \vdash A_2}\,\dot{\pi}_2^{\wedge}
$$

Truth for conjunction being defined by

$$
\mathcal{M} \vDash_\sigma^e A_1 \dot{\wedge} A_2 \;\triangleq\; \mathcal{M} \vDash_\sigma^e A_1 \wedge \mathcal{M} \vDash_\sigma^e A_2
$$

a case is added for conjunctive formulae in each direction of the proof of $\mathcal{M}_0 \vDash_\sigma A \Leftrightarrow A[\sigma] \in \mathcal{S}_\omega$ as follows:

$$
\begin{array}{lll}
\downarrow_\sigma^{A_1 \dot{\wedge} A_2} \;(m_1, m_2) & \triangleq & \dot{PAIR}\,(\downarrow_\sigma^{A_1} m_1, \downarrow_\sigma^{A_2} m_2) \\
\uparrow_\sigma^{A_1 \dot{\wedge} A_2} \;q & \triangleq & (\uparrow_\sigma^{A_1}(\dot{PROJ}_1^{\wedge}\, q), \uparrow_\sigma^{A_2}(\dot{PROJ}_2^{\wedge}\, q))
\end{array}
$$

where the following combinators lift inference rules on $\mathcal{S}_\omega$:

$$
\begin{array}{lll}
\dot{PAIR}(q_1, q_2) & \triangleq & dest\, share(q_1, q_2)\, as\, (n, \Gamma, f, p_1, p_2)\, in\, (n, \Gamma, f, \dot{pair}(p_1, p_2)) \\
\dot{PROJ}_i^{\wedge}(n, \Gamma, f, p) & \triangleq & (n, \Gamma, f, \dot{\pi}_i^{\wedge}\, p)
\end{array}
$$

In particular, there is no need to consider conjunctive formulae in the enumeration.

2.6. Extension to disjunction. Taking inspiration from works on normalisation-by-evaluation in the presence of disjunction (e.g. [1, 2]), we give a proof for disjunction that relies on the following arithmetical generalised form of Double Negation Shift introduced[30] in [50]:

$$\mathsf{DNS}^{\forall} \qquad \forall n\,((\mathcal{A}(n) \Rightarrow C) \Rightarrow C) \Rightarrow (\forall n\,\mathcal{A}(n) \Rightarrow C) \Rightarrow C$$

for C a Σ^0_1-formula and $\mathcal{A}$ arbitrary.

The proof however requires a significant change: instead of proving $\mathcal{M}_0 \vDash^e_\sigma A \Leftrightarrow A[\sigma] \in \mathcal{S}_\omega$, we prove:

$$\begin{array}{lcl} \downarrow^A_\sigma & : & \mathcal{M}_0 \vDash^e_\sigma A \Rightarrow A[\sigma] \in \mathcal{S}_\omega \\ \uparrow'^A_\sigma & : & A[\sigma] \in \mathcal{S}_\omega \Rightarrow \mathsf{KONT}(\mathcal{M}_0 \vDash^e_\sigma A) \end{array}$$

where $\mathsf{KONT}(\mathcal{A})$, a continuation monad[31], is defined by:

$$\mathsf{KONT}(\mathcal{A}) \triangleq (\mathcal{A} \Rightarrow \mathcal{T}_0, \dot{\neg}A_0 \vdash \dot{\bot}) \Rightarrow \mathcal{T}_0, \dot{\neg}A_0 \vdash \dot{\bot}$$

Let us assume the object language equipped with the following rules for disjunction:

$$\frac{\Gamma \vdash A_1}{\Gamma \vdash A_1 \dot{\vee} A_2}\,\dot{inj}_1 \qquad \frac{\Gamma \vdash A_2}{\Gamma \vdash A_1 \dot{\vee} A_2}\,\dot{inj}_2 \qquad \frac{\Gamma \vdash A_1 \dot{\vee} A_2 \quad \Gamma \vdash A_1 \dot{\Rightarrow} B \quad \Gamma \vdash A_2 \dot{\Rightarrow} B}{\Gamma \vdash B}\,\dot{case}$$

Let us also consider the following lifting of the inference rules to provability in $\mathcal{S}_\omega$:

$$\begin{array}{lll} \dot{INJ}_i : & A_i \in \mathcal{S}_\omega & \Rightarrow\; A_1 \dot{\vee} A_2 \in \mathcal{S}_\omega \\ \dot{INJ}_i & (n, \Gamma, f, p) & \triangleq\; (n, \Gamma, f, \dot{inj}_i\, p) \end{array}$$

$$\begin{array}{lll} \dot{CASE} : & \begin{pmatrix} A \dot{\vee} B \in \mathcal{S}_\omega \\ A \dot{\Rightarrow} C \in \mathcal{S}_\omega \\ B \dot{\Rightarrow} C \in \mathcal{S}_\omega \end{pmatrix} & \Rightarrow\; C \in \mathcal{S}_\omega \\ \dot{CASE} & (q, q_1, q_2) & \triangleq\; \begin{array}{l} dest\; share_3(q, q_1, q_2)\; as\; (n, \Gamma, f, p, p_1, p_2)\; in \\ (n, \Gamma, f, \dot{case}\; p\; p_1\; p_2) \end{array} \end{array}$$

where we needed the following three-part variant *share₃* of *share*:

$$\begin{array}{lll} share_3 : A_1 \in \mathcal{S}_\omega \wedge A_2 \in \mathcal{S}_\omega \wedge A_3 \in \mathcal{S}_\omega & \Rightarrow & \exists n \exists \Gamma (\Gamma \subset \mathcal{S}_n \wedge \Gamma \vdash A_1 \wedge \Gamma \vdash A_2 \wedge \Gamma \vdash A_3) \\ share_3 \quad (q_1, q_2, (n, \Gamma, f, p_3)) & \triangleq & \begin{array}{l} dest\; share(q_1, q_2)\; as\; (n', \Gamma', f', p_1, p_2)\; in \\ dest\; share((n', \Gamma', f', p_2), (n, \Gamma, f, p_3)) \\ as\; (n'', \Gamma'', f'', p_2', p_3) \\ in\; (n'', \Gamma'', f'', \dot{weak}(incl'^{\Gamma',\Gamma}_1, p_1), p_2', p_3) \end{array} \end{array}$$

Truth for disjunction being defined by

$$\mathcal{M} \vDash^e_\sigma A_1 \dot{\vee} A_2 \;\triangleq\; \mathcal{M} \vDash^e_\sigma A_1 \vee \mathcal{M} \vDash^e_\sigma A_2$$

we show the extended reification below where *case* does a case analysis on the form $inj_1(m)$ or $inj_2(m)$ of a proof of disjunction in the typed λ-calculus which we use to represent our metalanguage.

[30] In the presence of Markov's principle, this generalised form of DNS is equivalent to the usual form. The interest of the generalised form is precisely that it can be used in situations which would have required Markov's principle without requiring Markov's principle explicitly.

[31] Note that $\mathsf{KONT}(\mathcal{M}_0 \vDash^e_\sigma A)$ is actually the same as $\mathcal{M}_0 \vDash^e_\sigma \dot{\neg}\dot{\neg}A$ but we shall use KONT also on formulae which are not of the form $\mathcal{M}_0 \vDash^e_\sigma A$.

$$
\begin{array}{llll}
\downarrow^{A}_{\sigma} : & \mathcal{M}_0 \vDash^{e}_{\sigma} A & \Rightarrow & A[\sigma] \in \mathcal{S}_\omega \\
\downarrow^{P(\vec{t})}_{\sigma} & m & \triangleq & m \\
\downarrow^{\dot\bot}_{\sigma} & m & \triangleq & B\dot{O}T(m) \\
\downarrow^{A \dot\Rightarrow B}_{\sigma} & m & \triangleq & A\dot{X}_{A[\sigma]\dot\Rightarrow B[\sigma]}(kont'^{A\dot\Rightarrow B}_{\sigma}(m)) \\
\downarrow^{\dot\forall x A}_{\sigma} & m & \triangleq & AP\dot{P}^{\dot\Rightarrow}(A\dot{X}_{(\dot\forall x A)[\sigma]}, \downarrow^{A}_{\sigma,x\leftarrow x_n}(m\, x_n)) \ \textit{where}\ 2n = \ulcorner(\dot\forall x A)[\sigma]\urcorner \\
\downarrow^{A_1 \dot\wedge A_2}_{\sigma} & (m_1, m_2) & \triangleq & PA\dot{I}R\,(\downarrow^{A_1}_{\sigma} m_1, \downarrow^{A_2}_{\sigma} m_2) \\
\downarrow^{A_1 \dot\vee A_2}_{\sigma} & m & \triangleq & \textit{case } m \textit{ of } [inj_1(m_1) \mapsto I\dot{N}J_1(\downarrow^{A_1}_{\sigma} m_1) \\
 & & & \qquad\qquad \mid inj_2(m_2) \mapsto I\dot{N}J_2(\downarrow^{A_2}_{\sigma} m_2)]
\end{array}
$$

Note that this extended reification is unchanged except for the replacement of $kont^{A\dot\Rightarrow B}_{\sigma}$ by $kont'^{A\dot\Rightarrow B}_{\sigma}$ so as to take into account the use of KONT in $\uparrow'$:

$$kont'^{A\dot\Rightarrow B}_{\sigma}(m)(r) \triangleq (\uparrow'^{A}_{\sigma}\ (PR\dot{O}J^{\dot\Rightarrow}_1\ (r)))\,(m' \mapsto \textit{flush}\,(AP\dot{P}^{\dot\Rightarrow}\,(PR\dot{O}J^{\dot\Rightarrow}_2\ (r), \downarrow^{B}_{\sigma}\ (mm'))))$$

Before giving the modified reflection proof, we need to prove a form of ex falso quodlibet deriving the truth of any formula A from any inconsistency $\mathcal{T}_0, \dot\neg A_0 \vdash \bot$. This is a standard proof by induction on A:

$$
\begin{array}{llll}
EFQ^{A}_{\sigma} : & \mathcal{T}_0, \dot\neg A_0 \vdash \bot & \Rightarrow & \mathcal{M}_0 \vDash^{e}_{\sigma} A \\
EFQ^{P(\vec{t})}_{\sigma} & (\Gamma, g, p) & \triangleq & (0, (\Gamma, \dot\neg A_0), I_0(g), \textit{efq}\ p) \\
EFQ^{\dot\bot}_{\sigma} & (\Gamma, g, p) & \triangleq & (\Gamma, g, p) \\
EFQ^{A\dot\Rightarrow B}_{\sigma} & (\Gamma, g, p) & \triangleq & m \mapsto EFQ^{B}_{\sigma}(\Gamma, g, p) \\
EFQ^{\dot\forall x A}_{\sigma} & (\Gamma, g, p) & \triangleq & t \mapsto EFQ^{A}_{\sigma,x\leftarrow t}(\Gamma, g, p) \\
EFQ^{A_1 \dot\wedge A_2}_{\sigma} & (\Gamma, g, p) & \triangleq & (EFQ^{A_1}_{\sigma}(\Gamma, g, p), EFQ^{A_2}_{\sigma}(\Gamma, g, p)) \\
EFQ^{A_1 \dot\vee A_2}_{\sigma} & (\Gamma, g, p) & \triangleq & inj_1\,(EFQ^{A_1}_{\sigma}(\Gamma, g, p))
\end{array}
$$

where we may notice in passing that an arbitrary choice is made in the disjunction case.

We need the shift of double negation with respect to implication to formulae expressing truth. Ex falso quodlibet being obtained on formulae expressing truth with respect to possibly-exploding models, the proof is standard:

$$
\begin{array}{lll}
\mathsf{DNS}^{\Rightarrow} & : & (\mathcal{A} \Rightarrow \mathsf{KONT}(\mathcal{M}_0 \vDash^{e}_{\sigma} B)) \ \Rightarrow\ \mathsf{KONT}(\mathcal{A} \Rightarrow \mathcal{M}_0 \vDash^{e}_{\sigma} B) \\
\mathsf{DNS}^{\Rightarrow} & \triangleq & H \mapsto K \mapsto K\,(m_A \mapsto EFQ^{B}_{\sigma}\,(H\ m_A\ (m_B \mapsto K\ (m'_A \mapsto m_B))))
\end{array}
$$

We also need the shift of double negation with respect to conjunction. The intuitionistic proof is easy:

$$
\begin{array}{lll}
\mathsf{DNS}^{\wedge} & : & \mathsf{KONT}(\mathcal{A}_1) \wedge \mathsf{KONT}(\mathcal{A}_2) \Rightarrow \mathsf{KONT}(\mathcal{A}_1 \wedge \mathcal{A}_2) \\
\mathsf{DNS}^{\wedge} & \triangleq & (H_1, H_2) \mapsto K \mapsto H_1\,(m_1 \mapsto H_2\,(m_2 \mapsto K\,(m_1, m_2)))
\end{array}
$$

We are now ready to reformulate reflection, including the case for disjunction:

$$
\begin{array}{llll}
\uparrow'^{A}_{\sigma} : & A[\sigma] \in \mathcal{S}_\omega & \rightarrow & \mathsf{KONT}\,(\mathcal{M}_0 \vDash^{e}_{\sigma} A) \\
\uparrow'^{P(\vec{t})}_{\sigma} & q & \triangleq & K \mapsto K\,q \\
\uparrow'^{\dot\bot}_{\sigma} & q & \triangleq & K \mapsto \textit{flush}\,q \\
\uparrow'^{A\dot\Rightarrow B}_{\sigma} & q & \triangleq & \mathsf{DNS}^{\Rightarrow}(m \mapsto \uparrow'^{B}_{\sigma}\ (AP\dot{P}^{\dot\Rightarrow}\,(q, \downarrow_A\ m))) \\
\uparrow'^{\dot\forall x A(x)}_{\sigma} & q & \triangleq & \mathsf{DNS}^{\forall}\,(t \mapsto \uparrow'^{A}_{\sigma,x\leftarrow t}\ (AP\dot{P}^{\dot\forall}(q, t))) \\
\uparrow'^{A_1 \dot\wedge A_2}_{\sigma} & q & \triangleq & \mathsf{DNS}^{\wedge}(\uparrow'^{A_1}_{\sigma}\ (PR\dot{O}J^{\wedge}_1\ q), \uparrow'^{A_2}_{\sigma}\ (PR\dot{O}J^{\wedge}_2\ q)) \\
\uparrow'^{A_1 \dot\vee A_2}_{\sigma} & q & \triangleq & K \mapsto \textit{flush}\,(CA\dot{S}E(q, A\dot{X}_{\dot\neg A_1}(kont^{A_1 \dot\vee A_2}_{\sigma,1}(K)), \\
 & & & \qquad\qquad\qquad A\dot{X}_{\dot\neg A_2}(kont^{A_1 \dot\vee A_2}_{\sigma,2}(K))))
\end{array}
$$

where, for K proving $\mathcal{M}_0 \vDash^e_{id} \dot{\neg}(A_1 \dot{\vee} A_2)$, the continuation $kont^{A_1 \dot{\vee} A_2}_{\sigma,i}(K)$ is defined by:

$$\begin{array}{lcl} kont^{A_1 \dot{\vee} A_2}_{\sigma,i}(K) & : & (\mathcal{S}_{\lceil \dot{\neg} A_i[\sigma] \rceil}, \dot{\neg} A_i[\sigma] \vdash \dot{\bot}) \Rightarrow (\mathcal{T}_0, \dot{\neg} A_0 \vdash \dot{\bot}) \\ kont^{A_1 \dot{\vee} A_2}_{\sigma,i}(K) & \triangleq & r \mapsto (\uparrow'^{A_i}_{\sigma} (P\dot{R}OJ^{\dot{\Rightarrow}}_1 (r)))\,(m \mapsto K\,(inj_i\, m)) \end{array}$$

The proof of $classic_0$ follows a different pattern than the one without KONT. For it and for the proof of $init_0$, we use again $\mathsf{DNS}^{\forall}$ to distribute the quantification over the axioms of the theory:

$$\begin{array}{lcl} classic_0 & : & \mathsf{KONT}(\forall A\, ((\mathcal{M}_0 \vDash^e_{id} \dot{\neg}\dot{\neg} A) \Rightarrow (\mathcal{M}_0 \vDash^e_{id} A))) \\ classic_0 & \triangleq & \mathsf{DNS}^{\forall}\, (A \mapsto \mathsf{DNS}^{\Rightarrow}(m \mapsto m)) \end{array}$$

$$\begin{array}{lcl} init_0 & : & \mathsf{KONT}(\forall B \in \mathcal{T}_0\, \mathcal{M}_0 \vDash^e_{id} B) \\ init_0 & \triangleq & \mathsf{DNS}^{\forall}\, (B \mapsto \mathsf{DNS}^{\Rightarrow}(h \mapsto \uparrow'^{B}_{id} (0, (B, \dot{\neg} A_0), I_0(J_{cons}(J_{base}, h)), \dot{a}x_1))) \end{array}$$

Finally, the proof of completeness also needs to chain continuations:

$$\begin{array}{lcl} completeness & : & \forall \mathcal{M} \forall \sigma\, (\mathcal{M} \vDash^e_{\sigma} Classic \Rightarrow \mathcal{M} \vDash^e_{\sigma} \mathcal{T}_0 \Rightarrow \mathcal{M} \vDash^e_{\sigma} A_0) \Rightarrow \mathcal{T}_0 \vdash A_0 \\ completeness & \triangleq & \psi \mapsto DN\dot{A}BS\, (completeness'\, \psi) \end{array}$$

where

$$\begin{array}{l} completeness' : \forall \mathcal{M} \forall \sigma\, (\mathcal{M} \vDash^e_{\sigma} Classic \Rightarrow \mathcal{M} \vDash^e_{\sigma} \mathcal{T}_0 \Rightarrow \mathcal{M} \vDash^e_{\sigma} A_0) \Rightarrow \dot{\neg} A_0, \mathcal{T}_0 \vdash \dot{\bot} \\ completeness' \triangleq \psi \mapsto classic_0\, (c \mapsto init_0\, (i \mapsto flush\, (AP\dot{P}^{\dot{\Rightarrow}}\, (A\dot{X}^0_{\dot{\neg} A_0}, \downarrow^{A_0}_{id} (\psi\, \mathcal{M}_0\, id\, c\, i))))) \end{array}$$

An interesting remark is that $\mathcal{T}_0, \dot{\neg} A_0 \vdash \dot{\bot}$ intuitionistically implies $\mathsf{KONT}(\bot)$, so, by using KONT, reflection for $\bot$ does not need any more to consider possibly-exploding model in order to be intuitionistically valid. The need for possibly-exploding models (or Markov's principle) shows up instead in proving $EFQ^{\bot}_{\sigma} : \mathcal{T}_0, \dot{\neg} A_0 \vdash \dot{\bot} \Rightarrow \mathcal{M}_0 \vDash^e_{\sigma} \dot{\bot}$ which is in turn required to prove our specific version of $\mathsf{DNS}^{\Rightarrow}$, itself used to prove reflection for the connective $\dot{\Rightarrow}$, as well as to prove $classic_0$ and $init_0$.

We conjecture that the use of $\mathsf{DNS}^{\forall}$ is necessary, that is that the statement of completeness in the presence of disjunction implies $\mathsf{DNS}^{\forall}$ for $\exists$-free formulae. For instance, Kirst [57] proved in the slightly different context of intuitionistic epistemic logic that completeness with respect to (non-exploding) Kripke semantics implies the following principle:

$$\neg\neg \forall n\, (\neg \mathcal{A}(n) \vee \neg\neg \mathcal{A}(n))$$

which in turn is equivalent to:

$$\forall n\, \neg\neg\, (\neg \mathcal{A}(n) \vee \neg \mathcal{B}(n)) \Rightarrow \neg\neg \forall n\, (\neg \mathcal{A}(n) \vee \neg \mathcal{B}(n))$$

Note also that like Markov's principle, $\mathsf{DNS}^{\forall}$ preserves the witness and disjunction properties of intuitionistic logic, so they are *in this sense* intuitionistically valid. Also, to compute with $\mathsf{DNS}^{\forall}$, bar recursion [78] or delimited continuations [50] can be used.

2.7. Extension to existential quantification. The situation for existential quantification is simpler than for disjunction and modifying the statement of reflection is not necessary. The idea is to consider an enumeration of formulae which takes existential formulae into account, then to add a clause to the definition of $\Gamma \subset \mathcal{S}_n$ similar to the one for universal quantification, using Henkin's axiom $\dot{\exists} y\, A(y) \dot{\Rightarrow} A(x)$ for x taken fresh in the finite set of formulae coming before $\dot{\exists} y\, A(y)$ in the enumeration[32]. Reification is direct, using the witness coming from the proof of truth as a witness for the proof of derivability. For reflection, the idea is to combine a proof of $(\exists y\, A)[\sigma] \in \mathcal{S}_\omega$ with the proof of $(\dot{\exists} y\, A)[\sigma] \dot{\Rightarrow} A[\sigma, y \leftarrow x]$ available at some level $\mathcal{S}_n$ to get a proof of $A[\sigma, y \leftarrow x] \in \mathcal{S}_\omega$, then a proof of $\mathcal{M} \vDash^e_{\sigma, y \leftarrow x} A$, thus a proof [33] of $\mathcal{M} \vDash^e_\sigma \exists y\, A$.

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