

# Sharing in the weak lambda-calculus

Tomasz Blanc  
Jean-Jacques Lévy  
Luc Maranget

INRIA Rocquencourt

Dec 19, 2005

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 $\rightarrow$   
 $M[x := N]$   
 $+$   
 $\delta M M \rightarrow M$   
 $=$   
non confluent



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"Retomber dans la **clope**, JAMAIS!!"

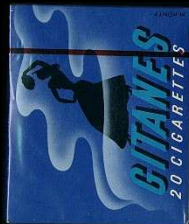
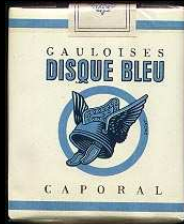
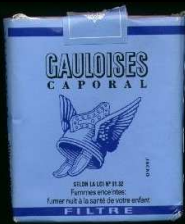
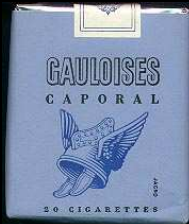
Et sinon ca va sans clope ?

<http://sos-cloppes.over-blog.com/>

cette saloperie de clope

Voila comment j'ai réussi à stopper la clop

il clope comme un pompier



t'as pas une clope ?

**FUMER UNE CLOPE**

ca vaut pas un clope. Des clopes !

des clopinettes.



## Laboratoire d'Analyse et de Traitement Informatique de la Langue Française

### Trésor de la Langue Française informatisé (version simplifiée)

Nouvelle recherche



#### Signification des couleurs

Mot  
recherché

Expressions  
ou locutions

Définitions

**CLOP(E)**, (**CLOP**, **CLOPE**) subst. masc.

Arg. **Mégot de cigare ou de cigarette**. *Jeter, ramasser, fumer un clope* :

- Ô mon vieux Maroni, ô Cayenne la douce!  
Je vois les corps penchés de quinze à vingt fagots  
Autour du mino blond qui fume les mégots  
Crachés par les gardiens dans les fleurs et la mousse.  
Un **clop** mouillé suffit à nous désoler tous.  
GENËT, *Poèmes*, Le condamné à mort, 1948, p. 23.

– P. ext. **Cigarette**. *Le Nantais posa son clope dans le cendrier* (A. LE BRETON, *Razzia sur la chnouf*, 1954, p. 32)

– Loc. **Des clopes**. **Rien** (cf. ESN. 1966).

**Étymol. et Hist.** Apr. 1900 *clope* « mégot de cigare ou de cigarette » (Notes manuscrites ajoutées sur les feuillets des notes de Nouguiér, p. 72); 1925 loc. *des clopes* « rien » (expr. pop. *béqueter des clopes* « jeûner » d'apr. ESN.); 1942 « cigarette » (maquisards d'apr. ESN. : se rouler un **clope**); 1947 *clap* « mégot » (L. STOLLÉ, *Douze récits hist. racontés en arg.*, p. 5). Orig. inconnue (*FEW* t. 21, p. 501; ESN.). **Fréq. abs. littér.** *Clop* : 1.



- 1 The weak  $\lambda$ -calculus
- 2 Properties of the weak  $\lambda$ -calculus
- 3 Sharing in the  $\lambda$ -calculus
- 4 Sharing in the weak  $\lambda$ -calculus
- 5 Sharing of subterms
- 6 Conclusion





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- $\lambda$ -calculus without the  $\xi$ -rule

$$(\xi) \frac{M \rightarrow N}{\lambda x.M \rightarrow \lambda x.N}$$

- is not confluent

$$\begin{array}{ccc}
 (\lambda x.\lambda y.M)N & \longrightarrow & (\lambda x.\lambda y.M)N' \\
 \downarrow & & \downarrow \\
 (\lambda y.M[[x \setminus N]]) & & (\lambda y.M[[x \setminus N']])
 \end{array}$$

- Our objectives :
    - find a confluent extension of the weak  $\lambda$ -calculus,
    - re-study standard properties (FD, standardization, etc),
    - find a theory of sharing in this calculus
- [Wadsworth, Shivers-Wand]



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- weakening the  $\xi$ -rule :

$$(\xi') \frac{M \xrightarrow{R} N \quad x \notin R}{\lambda x.M \xrightarrow{R} \lambda x.N}$$

( $R$  is the redex contracted in  $M \xrightarrow{R} N$ )

- redexes with free variables not bound in  $M$  can be contracted
- now

$$\begin{array}{ccc} (\lambda x.\lambda y.M)N & \longrightarrow & (\lambda x.\lambda y.M)N' \\ \downarrow & & \downarrow \\ (\lambda y.M[x \setminus N]) & \longrightarrow & (\lambda y.M[x \setminus N']) \end{array}$$



- $\lambda$ -terms

$$M, N ::= x \mid MN \mid \lambda x.M$$

- $\beta$ -reduction is

$$(\beta) \quad R = (\lambda x.M)N \xrightarrow{R} M[[x \setminus N]]$$

- Substitution  $M[[x \setminus N]]$  defined as usual :

$$\begin{aligned}x[[x \setminus P]] &= N \\y[[x \setminus P]] &= y \\(MN)[[x \setminus P]] &= M[[x \setminus P]] N[[x \setminus P]] \\(\lambda y.M)[[x \setminus P]] &= \lambda y.M[[x \setminus P]] \quad (x \neq y, y \notin P)\end{aligned}$$

- context rules

$$(\nu) \frac{M \xrightarrow{R} M'}{MN \xrightarrow{R} M'N}$$

$$(\mu) \frac{N \xrightarrow{R} N'}{MN \xrightarrow{R} MN'}$$

$$(\xi') \frac{M \xrightarrow{R} M' \quad x \notin R}{\lambda x.M \xrightarrow{R} \lambda x.M'}$$

- extra rules

- unlabelling

$$(w) \frac{M \xrightarrow{R} N}{M \rightarrow N}$$

- $M \twoheadrightarrow N$  for transitive and reflexive closure



**Theorem 1** [Church-Rosser] The weak  $\lambda$ -calculus is confluent.

Proof: Standard Tait–Martin-Lof proof.  $\square$

- residuals of disjoint redexes are disjoint.
  - $(\lambda x.lx)(Jy)$  with  $I, J = \lambda x.x$ .
  - In strong  $\lambda$ -calculus, the two disjoint  $lx$  and  $Jy$  redexes have nested residuals :

$$(\lambda x.lx)(Jy) \rightarrow I(Jy)$$

- impossible in weak  $\lambda$ -calculus.
- Finite developments theorem is easy to prove.



- standard reductions

$$M = M_0 \rightarrow M_1 \rightarrow \dots M_n = N \quad (n \geq 0)$$

$\forall i. \forall j. 0 \leq i < j < n$ , then  $R_j$  is not a residual of a redex internal to or to the left of the  $R_i$ .

- We write  $M \xrightarrow[\text{st}]{} N$

**Theorem 2** [Standardization] If  $M \rightarrow M'$ , then  $M \xrightarrow[\text{st}]{} M'$ .

- Normalization strategy to the “best” normal form (normal reduction is weak until abstractions).



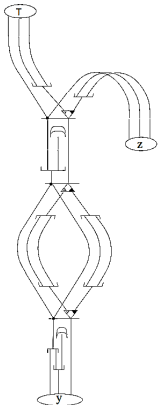
- weak explicit substitutions with closures
- Hindley's rule

$$(\sigma) \frac{N \rightarrow N'}{M[[x \setminus N]] \rightarrow M[[x \setminus N']]}$$

- computational monads
- Ariola, et al ; Launchbury
- explicit substitutions (not confluent, non normalizable)
- classic  $\lambda$ -calculus (confluent, normalizable, complex theory of sharing)



- difficult in classical  $\lambda$ -calculus  
⇒ interaction nets + geometry of interaction



- not elementary recursive



Take  $(\lambda x.k(xa)(xb))(\lambda y.(ly)) \rightarrow k(\bullet a)(\bullet b)$  where  $\bullet = \lambda y.(ly)$

- in the classical  $\lambda$ -calculus,
  - sharing is complex because of sharing of functions
  - sharing of subcontexts
  - sharing of boxes
- in weak  $\lambda$ -calculus,
  - one cannot contract redexes whose free variables are bound in surrounding context
  - sharing of subterms
  - sharing of trees



## Sharing in the $\lambda$ -calculus (3/5)

- find a confluent theory of sharing
- sharing = labelling  
⇒ find **a confluent labelled**  $\lambda$ -calculus.



Terms :

$U, V ::= \alpha : X$  labeled term

$X, Y ::= S \mid U$  clipped or labeled term

$S, T ::= x \mid UV \mid \lambda x. U$  clipped term

$\alpha, \beta ::= a \mid [\alpha'] \mid [\alpha']$  labels

$\alpha', \beta' ::= \alpha_1 \alpha_2 \cdots \alpha_n \quad (n > 0)$  compound labels

Reduction

$$(l) \quad (\alpha' \cdot \lambda x. U) V \rightarrow [\alpha'] : U \llbracket x \setminus [\alpha'] : V \rrbracket$$

where

$$\alpha_1 \alpha_2 \cdots \alpha_n \cdot S = \alpha_1 : \alpha_2 : \cdots : \alpha_n : S$$



## Context rules

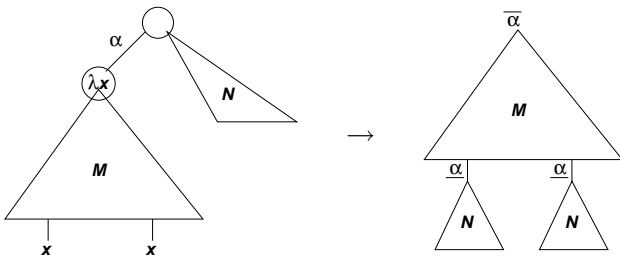
$$(\nu) \frac{U \rightarrow U'}{UV \rightarrow U'V}$$

$$(\lambda) \frac{X \rightarrow X'}{\alpha : X \rightarrow \alpha : X'}$$

$$(\mu) \frac{V \rightarrow V'}{UV \rightarrow UV'}$$

$$(\xi) \frac{U \rightarrow U'}{\lambda x. U \rightarrow \lambda x. U'}$$

## Graphically





Terms :

$U, V$	::= $\alpha : X$	labeled term
$X, Y$	::= $S \mid U$	clipped or labeled term
$S, T$	::= $x \mid UV \mid \lambda x. U$	clipped term
$\alpha, \beta$	::= $\mathbf{a} \mid [\alpha'] \mid [\alpha'] \mid [\alpha', \beta] \mid \langle \alpha', \beta \rangle$	labels
$\alpha', \beta'$	::= $\alpha_1 \alpha_2 \cdots \alpha_n \quad (n > 0)$	compound labels

Reduction

$$(\ell) \quad R = (\alpha' \cdot \lambda x. U) V \xrightarrow{R} [\alpha'] : (\alpha' \otimes U) \llbracket x \setminus [\alpha'] : V \rrbracket$$

where

$$\alpha_1 \alpha_2 \cdots \alpha_n \cdot \mathbf{S} = \alpha_1 : \alpha_2 : \cdots \alpha_n : \mathbf{S}$$



# Sharing in the weak $\lambda$ -calculus (2/5)

## Context rules

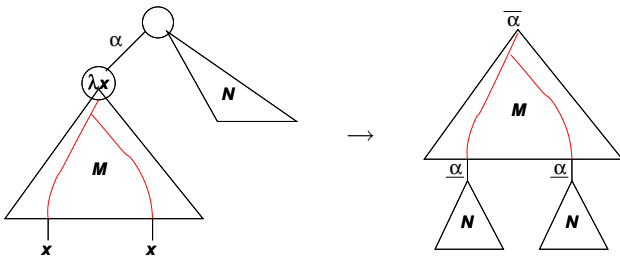
$$(\nu) \frac{U \xrightarrow{R} U'}{UV \xrightarrow{R} U'V}$$

$$(\lambda) \frac{X \xrightarrow{R} X'}{\alpha : X \xrightarrow{R} \alpha : X'}$$

$$(\mu) \frac{V \xrightarrow{R} V'}{UV \xrightarrow{R} UV'}$$

$$(\xi') \frac{U \xrightarrow{R} U' \quad x \notin R}{\lambda x. U \xrightarrow{R} \lambda x. U'}$$

## Graphically





## Diffusion

$$\alpha' \otimes X = X \text{ if } x \notin X$$

$$\alpha' \otimes x = x$$

$$\alpha' \otimes \lambda y. U = \lambda y. \alpha' \otimes U \text{ if } x \in \lambda y. U$$

$$\alpha' \otimes \beta : X = [\alpha', \beta] : \alpha' \otimes X \text{ if } x \in X$$

tagging

$$\alpha' \otimes UV = (\alpha' \otimes U \ \alpha' \otimes V) \text{ if } x \in U$$

$$\alpha' \otimes UV = (\langle \alpha', U \rangle \ \alpha' \otimes V) \text{ if } x \notin U \text{ and } x \in V$$

marking

$$\langle \alpha', \beta : X \rangle = \langle \alpha', \beta \rangle : X$$



Diffusion in  $R = (\alpha' \cdot \lambda x.U)V \xrightarrow{R} [\alpha'] : (\alpha' \otimes U)[x \setminus [\alpha'] : V]$

- “tagging” paths to occurrences of free variable  $x$ .
- “marking” redexes unleashed by reduction of  $R$ .
  - created redexes by contraction of  $R$  are tagged or marked by  $\alpha'$ . They can also contain  $[\alpha']$  or  $[\alpha']$ .
  - residual of redexes with name  $\alpha'$  are also named  $\alpha'$ .
  - “marking” is necessary in following example :  
 $R = (\lambda x.lx)y$ , where  $l = \lambda u.u$ .  
 Then  $lx$  is not a redex in  $(\lambda x.lx)y$ ,  
 but it becomes redex  $ly$  after contracting  $R$ .



**Lemma 1** If  $X \xrightarrow{R} X'$  and  $x \notin R$ , then  $\alpha' \otimes X \rightarrow \alpha' \otimes X'$

**Lemma 2** If  $U \rightarrow U'$ , then  $X[[x \setminus U]] \twoheadrightarrow X[[x \setminus U']]$

**Theorem 3** [Church-Rosser] The weak labeled  $\lambda$ -calculus is confluent.

Proof: By the Tait–Martin-Lof method.  $\square$



- $\lambda$ -terms are represented by dags,
- labels represent addresses in dags,
- at beginning no sharing, all addresses of subterms are distinct.

## Notation

*Init(U) when every subterm of U is labeled with a distinct letter (a, b, c, ...).*

**Invariant 1**  $\mathcal{P}(W)$  holds iff, for any couple of subterms  $\alpha: X$  and  $\beta: Y$  such that  $\alpha \simeq \beta$ , we have  $X = Y$ .

**Theorem 4** Let  $Init(U)$  and  $U \Longrightarrow V$ , then  $\mathcal{P}(V)$ .



# Sharing of subterms (2/4)

- $\alpha \simeq \beta$  when  $\alpha = \beta$  up to marking
- $U \xrightarrow{\alpha'} V$  when all redexes of name  $\alpha'$  are contracted in  $U$ , result is  $V$ .

where

$$a \simeq a$$

$$[\alpha'] \simeq [\alpha']$$

$$[\alpha'] \simeq [\alpha']$$

$$\beta \simeq \gamma \Rightarrow [\alpha', \beta] \simeq [\alpha', \gamma]$$

$$\beta \simeq \gamma \Rightarrow \langle \alpha', \beta \rangle \simeq \langle \alpha', \gamma \rangle$$

$$\beta \simeq \gamma \Rightarrow \beta \simeq \langle \alpha', \gamma \rangle$$

$$\beta \simeq \gamma \Rightarrow \langle \alpha', \beta \rangle \simeq \gamma$$

**Lemma 3** If  $X \xrightarrow{R} Y$  and redex  $S$  in  $Y$  is created by this reduction step, then  $name(R) \prec name(S)$ .

$$\alpha' \prec [\alpha']$$

$$\alpha' \prec [\alpha']$$

$$\alpha' \prec [\alpha', \beta]$$

$$\alpha' \prec \langle \alpha', \beta \rangle$$

$$\alpha' \prec \beta_i \Rightarrow \alpha' \prec \beta_1 \dots \beta_n$$

$$\alpha' \prec \beta' \prec \gamma' \Rightarrow \alpha' \prec \gamma'$$



## Sharing of subterms (3/4)

- interesting proof, with 4 invariants

**Invariant 2**  $\mathcal{Q}(W)$  holds iff we have  $\alpha' \not\prec \beta$  for every redex  $R$  with name  $\alpha'$  and any subterm  $\beta:X$  in  $W$ .

**Invariant 3**  $\mathcal{R}(W)$  holds iff for any clipped subterm  $UV$  in  $W$ , we have either  $U = a:X$ , or  $U = [\alpha', \beta]:X$ , or  $U = \langle \alpha', \beta \rangle :X$ .

**Invariant 4**  $\mathcal{S}(W)$  holds iff, for any application subterms  $\beta:(\alpha:X)U$  and  $\gamma:(\alpha:Y)V$ , we have  $\beta \simeq \gamma$ .

**Lemma 4** If  $\mathcal{Q}(W)$  and  $W \xrightarrow{\gamma'} W'$ , then  $\mathcal{Q}(W')$ .

**Lemma 5** If  $\mathcal{R}(W)$  and  $W \rightarrow W'$ , then  $\mathcal{R}(W')$ .

**Lemma 6** If  $\mathcal{P}(W) \wedge \mathcal{Q}(W) \wedge \mathcal{R}(W) \wedge \mathcal{S}(W)$  and  $W \xrightarrow{\gamma'} W'$ , then  $\mathcal{S}(W')$ .

**Lemma 7** If  $\mathcal{P}(W) \wedge \mathcal{Q}(W) \wedge \mathcal{R}(W) \wedge \mathcal{S}(W)$  and  $W \xrightarrow{\gamma'} W'$ , then  $\mathcal{P}(W')$ .



- labeled  $\lambda$ -calculus corresponds to Wadsworth's PhD (ch.4)  
2nd method
  - diffusion = copying
  - labels = adresses
  - calculus is confluent
- how to check  $x \in U$  efficiently ?
  - Shivers-Wand's method (bottom-up copying from bound variables to root of function bodies.
  - our method models slightly more shared strategy since not recursively copying binders met on path to root of function bodies.
  - compiling this check is not easy since sets of variables may change during computation.
  - similar to full laziness (PJ, Hugues), but without super combinators.



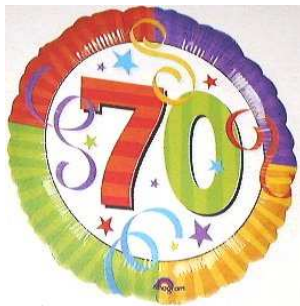
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- re-do all theory of optimal reductions,
- links with supercombinators and other compiler techniques,
- weak  $\lambda$ -calculus deserves a theory
- theory simpler than for TRS

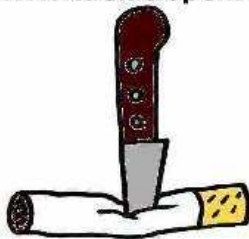


## OBJECTIVE





[www.tuelaclope.fr.st](http://www.tuelaclope.fr.st)



Tue la clope  
avant qu'elle ne  
te tue!