Generalized Finite Developments

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Abstract. The Finite Development theorem (FD) is a fundamental theorem in the theory of the syntax of the lambda-calculus. It gives sense to parallel reductions by stating that one can contract any given set of (possibly nested) redexes in any lambda term without looping and caring about the order in which these redexes are contracted. This theorem can be used to prove the Church-Rosser property, thus insuring determinism of reductions and uniqueness of normal forms. This paper explains how to extend the FD theorem to a finite number of creations of new redexes, i.e. redexes which do not exist in the initial term. This generalized theorem (gFD) also provides a proof technique to show the completeness of various reduction strategies. Finally it gives a natural intuition to the strong normalization property of the standard first-order typed lambda-calculus. The results in this article are not new, but were often mixed with other arguments; the aim of this paper is to stress on this sole gFD theorem.

1 Introduction

The basic operation of the lambda-calculus [6] is beta-reduction

$$(\lambda x.A)B \rightarrow A\{x := B\}$$

contracting any redex of the form $(\lambda x.A)B$ into its contractum $A\{x := B\}$ where every occurrence of the free variable x in A is replaced by B. Intuitively $\lambda x.A$ is a function of x producing the term A and beta-reduction describes the passing of the argument B to the function $\lambda x.A$ as in programming languages [4, 13, 19]. The lambda-calculus enjoys the Church-Rosser property, also named full confluence or simply confluence, which ensures the determinacy of results for beta-reductions. For instance if we write $\Delta = \lambda x.xx$ and $I = \lambda x.x$, then we have

$$M = (Ia)(Ia) \leftarrow \Delta(Ia) \rightarrow \Delta a = N$$

and $M \rightarrow a(Ia) \rightarrow aa \leftarrow N$. The common reduct aa of M and N can only be reached by performing two steps from M since the Ia redex is duplicated in M. The proof of the Church-Rosser theorem relies on an inductive argument on the length of *parallel reductions* starting from a given initial term, namely reductions whose parallel steps are of the form

$$M \xrightarrow{\mathcal{F}} M'$$

simultaneously contracting a set $\mathcal{F}$ of redexes in M. For instance, in the previous example, we have

$$M \xrightarrow{\mathcal{F}} aa \leftarrow N$$

where $\mathcal{F}$ is the set of the two Ia redexes in M and the usual reduction step $\rightarrow$ is identified to the contraction of a singleton set of redexes.

The exact definition of these parallel steps is not straightforward. Tait and Martin-Lof [3] axiomatize an anonymous parallel step as the contraction of a given set of redexes in an inside-out way. Curry and Feys [7] use the finite development (FD) theorem. This latter method is more ambitious since it shows that the notion of parallel step is consistent whatever the order in which redexes are contracted. An easy corollary of the FD theorem is the so-called lemma of parallel moves stating that if $\mathcal{F}_1$ and $\mathcal{F}_2$ are two sets of redexes in the lambda-term A and if

$$M \xleftarrow{\mathcal{F}_1} A \xrightarrow{\mathcal{F}_2} N$$

there exists a lambda-term B such that

$$M \xrightarrow{\mathcal{F}_2'} B \xleftarrow{\mathcal{F}_1'} N$$

The proof of the lemma just considers developments of $\mathcal{F}_1 \cup \mathcal{F}_2$. Then full confluence of the lambda-calculus follows easily.

In this article, we look more closely to the FD theorem and want to exhibit the property which makes it hold. Traditionally, the FD theorem is divided into two parts. Let $\mathcal{F}$ be a set of redexes in a given term M. Firstly any reduction relative to $\mathcal{F}$ terminates; secondly all maximal reductions relative to $\mathcal{F}$ end with the same term. In the previous statement, we say that a reduction is relative to $\mathcal{F}$ iff it only contracts residuals of redexes of $\mathcal{F}$. The FD theorem means that the set of reductions relatives to $\mathcal{F}$ forms a *canonical system*, i.e. a set of confluent finite reductions. This situation is analogous to the one of the (first-order or higher-order) typed lambda-calculus where the set of all reductions from any given term also forms a canonical system. The difference between the typed lambda-calculus and relative reductions comes from the fact that in the latter case no redex is created. In the untyped lambda-calculus, strong normalization is not valid since reductions may be infinite as with $\Delta\Delta$ where a new redex is always created. But in the typed case, redexes may only be created in a limited form.

The main objective of this paper is to demonstrate that if there is no infinite chain of creations of redexes, the FD theorem may be extended. Therefore we will be able to characterize arbitrarily large canonical subsets of the reduction graph of any lambda-term in the untyped lambda-calculus. It will therefore enlighten strong normalization as the impossibility of producing infinite chains of redex creations.

Some technique will be necessary in stating this generalized FD theorem. The notion of residual of redexes is subtle in the lambda-calculus since disjoint redexes may have nested residuals (this is not happening in term rewriting systems). We will also need the notion of permutation equivalence for reductions in order to define redex families. In section 2, we introduce Hyland-Wadsworth lambda-calculus. In section 3, we review definitions of residuals and finite developments. In section 4, we define historical redexes and redex families. In section 5, we show the generalized FD theorem and apply it to several lambda-calculi. Section 5 is the conclusion.

2 Hyland-Wadsworth lambda-calculus

Hyland-Wadsworth's lambda-calculus has been used for characterizing Scott's D_∞ model with Bohm trees. Here we use this calculus as a tool for proving termination of the generalized FD theorem.

In this calculus, the set Λ_e of terms is the usual set of lambda-terms but with every subterm equipped with an integer exponent. Beta-conversion increments the exponents on borders of contractums. The rest of the calculus follows the traditional rules of the untyped lambda-calculus. We also assume that there is a global integer constant L which defines a lambda-calculus up-to-L, that we call in short the $hw(L)$-calculus, as follows:

m, n ::= integer number — exponents ($m, n \geq 0$)

$U, V ::= x^n \mid (UV)^n \mid (\lambda x.U)^n$ — terms

$((\lambda x.U)^n V)^m \to U\{x := V_{[n+1]}\}_{[n+1][m]}$ — beta-conversion when $n \leq L$

$$x^n_{[m]} = x^p \qquad (UV)^n_{[m]} = (UV)^p \qquad (\lambda x.U)^n_{[m]} = (\lambda x.U)^p$$

projection where $p = \lceil m, n \rceil$

$$x^n\{x := W\} = W_{[n]}$$
$$(UV)^n\{x := W\} = (U\{x := W\}\ V\{x := W\})^n$$
$$(\lambda y.U)^n\{x := W\} = (\lambda y.U\{x := W\})^n$$

substitution

The calculus presented here is a slight variant of the exact rules defined by Hyland and Wadsworth which followed rules of application in Scott's D_∞ model where application decrements exponents and a special rule exists for application of a function with a null exponent. Here, we notice that any term is in normal form in the $hw(-1)$ calculus. If $\Delta_n = (\lambda x.(x^0 x^0)^0)^n$ and $\Omega_n = (\Delta_n \Delta_n)^n$, we have Ω_n in normal form when $n > L$ and $\Omega_n \to \Omega_{n+1}$ when $n \leq L$.

We define the *degree* of a redex $R = ((\lambda x.U)^n V)^m$ to be the exponent n of its function part. We write $degree(R) = n$. In the up-to-L lambda-calculus, a redex is reducible if and only if its degree is not greater than L. Therefore by increasing the global constant L one gives more power to the calculus and reductions may go further. The reduction graph $\mathcal{R}_L(U)$ starting form a given term U get larger when L increases. We have

$$\mathcal{R}_0(U) \subset \mathcal{R}_1(U) \cdots \subset \mathcal{R}_L(U) \subset \mathcal{R}_{L+1}(U) \cdots$$

Theorem 1. *hw(L) is a confluent and strongly normalizable calculus.*

The proof is rather easy and follows Tait-Martin Lof (for confluence) and van Daalen (for strong normalization). Q.E.D.

We also call *canonical system* a calculus which enjoys both confluence and strong normalization. Therefore $hw(L)$ is a canonical system for any L. It can also be proved that $hw(\infty)$ is also confluent, but it may not terminate as in the usual lambda-calculus.

Let the forgetful function $||U||$ be the lambda term obtained by erasing all exponents in U. For a given M, we can consider the term U_0 with all exponents equal to zero such that $||U_0|| = M$ and we may wonder what these increasing chain of sets $\mathcal{R}_L(U_0)$ represent. They approximate the reduction graph $\mathcal{R}(M)$ of M with an increasing set of canonical calculi.

3 Residuals and finite developments

Beta-reduction is written $\rightarrow$. Several steps (maybe none) of beta-reduction are written $\twoheadrightarrow$. We give names ρ, σ, ... to reductions and we write $\rho : M \twoheadrightarrow N$ to specify the initial and final terms of reduction ρ. The reduction graph of any lambda term M is written $\mathcal{R}(M)$. If two reductions ρ and σ start from M, we call them coinitial. Similarly, they are cofinal if they end on the same term N. Redexes may be tracked over reductions. Let $\rho \in \mathcal{R}(M)$ be the following reduction from M to N

$$\rho : M = M_0 \stackrel{R_1}{\rightarrow} M_1 \stackrel{R_2}{\rightarrow} M_2 \stackrel{R_2}{\rightarrow} \cdots \stackrel{R_n}{\rightarrow} + M_n = N$$

contracting redex R_i in M_{i-1} at each step ($1 \leq i \leq n$). Let R be a redex in M. There could be several (may be none) residuals of R in N. We write $R \backslash \rho$ for the set of residuals of R in M by reduction $\rho \in \mathcal{R}(M)$. For instance, if we underline R and its residuals, we have

$$\begin{aligned} \rho &: \Delta(\underline{Ix}) \rightarrow (\underline{Ix})\ (\underline{Ix}) \\ \sigma &: \underline{\Delta(Ix)} \rightarrow \underline{\Delta x} \\ \tau &: \underline{\Delta(Ix)} \rightarrow (Ix)(Ix) \end{aligned}$$

In the first case, Ix is copied by reduction ρ and has two residuals; in the second case, $\Delta(Ix)$ is modified by σ but is not copied by the contraction of an internal redex, and there is a single residual; in the third case, there is no residual of the contracted redex, since it disappeared. To define residuals precisely needs some work. We can either speak of occurrences of subterms and consider occurrences of both R and the redex contracted R_1 as in Curry&Feys [7]. An alternative method is Barendregt's underlined method [3]. We will refer to both methods here, and will not define precisely residuals. Just notice that residuals depend upon the reduction and that residuals of a redex may differ if we consider distinct reductions between two same terms. For instance, take $I(Ix) \rightarrow Ix$; then residuals are not the same if one contracts the external or internal redex.

Residuals are defined in a similar way in the $hw(L)$-calculus, since exponents do not modify the structure of terms. We first notice that residuals of a redex keep the degrees in $hw(L)$.

Lemma 1. *If $\rho : U \twoheadrightarrow V$ and $R = ((\lambda x.A)^n B)^m$ is a redex in U. Let R' be a redex in V such that $R' \in R\backslash\rho$, then degree(R') = degree(R).*

We consider now the $hw(0)$-calculus in which we can only contract redexes of null degree. We also observe that there are only two cases for the creation of new redexes in the lambda-calculus. We can either pass an abstraction to the left of an application as in $(\lambda x. \cdots xB \cdots)(\lambda y.A)$ or we can create a redex upward as in $(\lambda x.A)BD$ when we have $(\lambda x.A)B \rightarrow \lambda y.C$. In both cases, in the $hw(L)$-calculus, we notice that the degree of the created redex is strictly greater than the one of the contracted one. This is due to the increment used in the projection rules around the contractum of the contracted redex.

Therefore in $hw(0)$-calculus, when $\rho : U \twoheadrightarrow V$, a redex in V with null degree is a residual of a redex in the initial term U.

We can now state the finite development theorem with the proof used in [16].

Theorem 2 (Finite developments+). *Let $\mathcal{F}$ be a set of redexes in M, consider relative reductions which contract only residuals of redexes in $\mathcal{F}$. Then:*

1. *there is no infinite reduction relative to $\mathcal{F}$;*
2. *the developments of $\mathcal{F}$ (maximal relative reductions) end all at the same term;*
3. *let R be a redex in M (maybe not in $\mathcal{F}$), then the residuals of R are the same by all developments of $\mathcal{F}$.*

The proof is quite simple when using the Hyland-Wadsworth calculus. Consider the term M_0 where all subterms are equipped with a null exponent except degrees of redexes not in $\mathcal{F}$ which are equipped with degrees 2, 3, 4, etc. Then reductions relative to $\mathcal{F}$ are exactly the one of M_0 in $hw(0)$, since redexes of null degree can only be residuals of redexes in $\mathcal{F}$. Now as $hw(0)$ is a canonical system, one concludes on items 1 and 2. Furthermore, new redexes in $hw(0)$ can only have degree 1, and therefore redexes of degrees 2, 3, 4, etc can only be residuals of redexes in M_0 (not in $\mathcal{F}$). As the normal form is unique in $hw(0)$, the residuals of redexes not in $\mathcal{F}$ are the same by any development of $\mathcal{F}$. Q.E.D.

One may notice the elegance of the previous proof since we escape the inspection of multiple cases by working with $hw(0)$ in which this inspection has been done in a systematic way within the confluence proof.

The third item of the FD theorem is often skipped. In fact, it is an important clause since it shows that the notion of residual is consistent with parallel steps. If we simultaneously contract a set $\mathcal{F}$ of redexes, we can speak of residuals of redexes without specifying the order in which redexes of $\mathcal{F}$ are contracted. We therefore write $R\backslash\mathcal{F}$ for the set of residuals of R by any (finite) development of $\mathcal{F}$ and we are also free of writing

$$M \xrightarrow{\mathcal{F}} N$$

for the relation linking M to N by a development of $\mathcal{F}$.

Going on with notations, when $\rho : M \twoheadrightarrow N$ and $\sigma : N \twoheadrightarrow P$, we write $\rho\sigma$ for the concatenation of these two reductions. Furthermore, we write o for the empty reduction of length 0. Therefore we have $\rho o = \rho = o\rho$. We also write $\mathcal{F}$ for the reduction consisting in a finite development of $\mathcal{F}$, and we define $\mathcal{F}_1 \sqcup \mathcal{F}_2$ as the two-step long reduction $\mathcal{F}_1 \sqcup \mathcal{F}_2 = \mathcal{F}_1(\mathcal{F}_2 \backslash \mathcal{F}_1)$. We finally forget braces when $\mathcal{F}$ is a singleton $\{R\}$. We can write $R\rho$ or ρR or $R \backslash S$ or $R \sqcup S$ or etc.

There are two well-known corollaries of the FD theorem which will be useful for introducing the notion of redex history.

Lemma 2 (Parallel moves). *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two sets of redexes in M. If $M_1 \overset{\mathcal{F}_1}{\leftarrow} M \overset{\mathcal{F}_2}{\rightarrow} M_2$, then there is a term N such that $M_1 \overset{\mathcal{G}_2}{\rightarrow} N \overset{\mathcal{G}_1}{\leftarrow} M_2$ with $\mathcal{G}_1 = \mathcal{F}_1 \backslash F_2$ and $\mathcal{G}_2 = \mathcal{F}_2 \backslash F_1$.*

With our new notations, an alternative statement of this lemma would be that $\mathcal{F}_1 \sqcup \mathcal{F}_2$ and $\mathcal{F}_2 \sqcup \mathcal{F}_1$ are cofinal. The proof is obvious by considering two developments of $\mathcal{F}_1 \cup \mathcal{F}_2$, one starting by contracting $\mathcal{F}_1$ and then $\mathcal{F}_2 \backslash \mathcal{F}_1$, the second contracting $\mathcal{F}_2$ and then $\mathcal{F}_1 \backslash \mathcal{F}_2$.

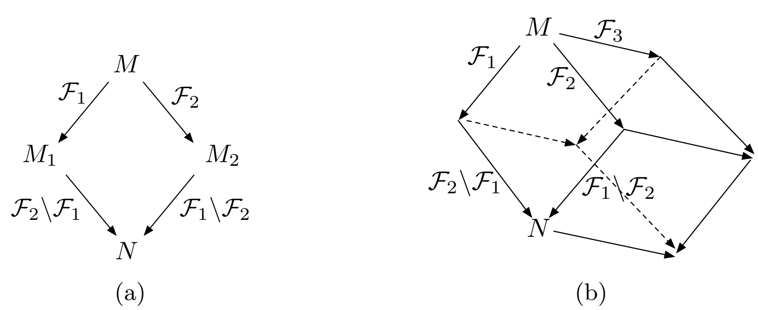


Fig. 1. (a) Parallel moves; (b) Cube lemma.

The lemma of parallel moves can also be named the square lemma, since a second corollary is the cube lemma which looks more closely to properties of residuals.

Lemma 3 (Cube lemma). *Let $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ be three sets of redexes in M. Then $\mathcal{F}_3 \backslash (\mathcal{F}_1 \sqcup \mathcal{F}_2) = \mathcal{F}_3 \backslash (\mathcal{F}_2 \sqcup \mathcal{F}_1)$.*

This lemma is straightforward when we notice that $\mathcal{F}_1 \sqcup \mathcal{F}_2$ and $\mathcal{F}_2 \sqcup \mathcal{F}_1$ are two developments of $\mathcal{F}_1 \cup \mathcal{F}_2$. This argument may be extended by permutation on $\mathcal{F}_2$ and $\mathcal{F}_3$, or $\mathcal{F}_3$ and $\mathcal{F}_1$. We therefore get a nice cube whose each face is an application of the parallel moves lemma.

4 Redex families

Redexes may be created by reductions when they are not residuals of redexes in the initial term. The notion of redex family aims to relate created redexes. As the reduction creating a redex interferes with its history, we define an *historical redex* (*hredex* in short) to be a pair $\langle \rho, R \rangle$ when $\rho : M \twoheadrightarrow N$ and R is a redex in N. Redexes in M with no history are of the form $\langle o, R \rangle$ where o is the empty reduction from M. But history of redexes has to be consistent with permutations of reduction steps, since the contraction of two independent redexes must be insensitive in the creation history of a redex.

Definition 1. *The* permutation equivalence $\sim$ *on reductions in* $\mathcal{R}(M)$ *starting from a given term* M *is defined inductively as follows:*

(i) $\mathcal{F}_1 \sqcup \mathcal{F}_2 \sim \mathcal{F}_2 \sqcup \mathcal{F}_1$
(ii) $\emptyset \sim o$ *and* $o \sim \emptyset$
(iii) $\rho \sim \sigma$ *implies* $\tau\rho \sim \tau\sigma$
(iv) $\rho \sim \sigma$ *implies* $\rho\tau \sim \sigma\tau$
(v) $\rho \sim \sigma \sim \tau$ *implies* $\rho \sim \tau$.

Two reductions are equivalent if they differ by applying the lemma of parallel moves. A reduction step contracting an empty set of redexes can be erased, i.e. it is equivalent to the empty reduction. Permutation equivalence is also extended by concatenation and by transitivity.

This relation is defined on parallel reductions, but it also relates regular reductions since their elementary steps can be considered as contracting singletons sets of redexes. In fact, the same relation can be generated by restricted the first case of the definition to be singleton sets.

Residuals of reductions can also be defined inductively as follows (see figure 2).

Definition 2. *Let* ρ *and* σ *be two reduction starting at* M. *The* residual $\rho\backslash\sigma$ of reduction ρ *by reduction* σ *is inductively defined by:*

1. *if* $\rho = \mathcal{F}$ *and* $\sigma = \mathcal{G}$, *then* $\rho\backslash\sigma = \mathcal{F}\backslash\mathcal{G}$
2. $\rho\backslash(\sigma_1\sigma_2) = (\rho\backslash\sigma_1)\backslash\sigma_2$
3. $(\rho_1\rho_2)\backslash\sigma = (\rho_1\backslash\sigma)(\rho_2(\sigma\backslash\rho_1)$

When ρ and σ are two coinitial reductions, we define $\rho \sqcup \sigma$ as for single-step reductions by $\rho \sqcup \sigma = \rho(\sigma\backslash\rho)$. We also write $\emptyset^k$ for k steps of empty-set contractions. The following properties hold for the permutation equivalence.

Lemma 4. *Let* ρ *and* σ *be two coinitial reductions* $(\rho, \sigma \in \mathcal{R}(M))$ *:*

(i) $\rho \sqcup \sigma \sim \sigma \sqcup \rho$
(ii) $\rho \sim \sigma$ *iff* $\forall \tau \in \mathcal{R}(M)\ \ \tau\backslash\rho = \tau\backslash\sigma$
(iii) $\rho \sim \sigma$ *iff* $\rho\backslash\sigma = \emptyset^k$ *and* $\sigma\backslash\rho = \emptyset^\ell$
(iv) $\rho\sigma \sim \rho\tau$ *iff* $\sigma \sim \tau$

The proofs use simple algebraic arguments. Clearly (i) is true by definition. The cube lemma gives (iia) which is that $\rho \sim \sigma$ implies $\tau\backslash\rho = \tau\backslash\sigma$. Then (iia) implies $(iiia)$ since $\emptyset^k = \rho\backslash\rho = \rho\backslash\sigma$ and $\sigma\backslash\rho = \sigma\backslash\sigma = \emptyset^\ell$. Conversely, when $\sigma\backslash\rho = \emptyset^\ell$, one

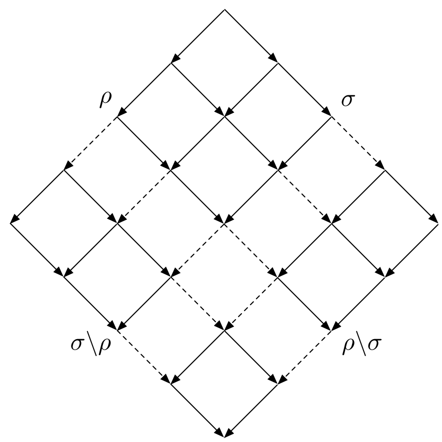
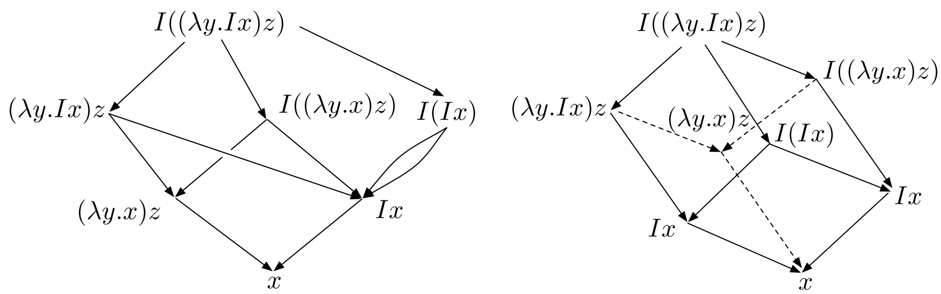


Fig. 2. $\rho\backslash\sigma$ is residual of reduction ρ by reduction σ.

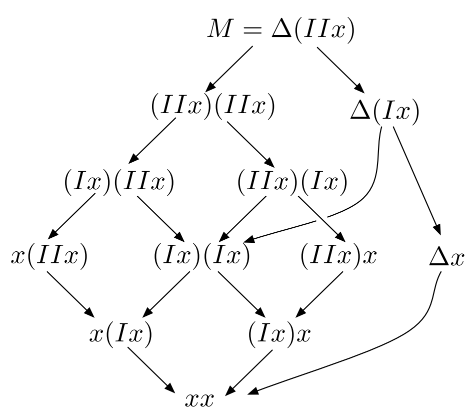
gets $\rho \sim \rho(\sigma\backslash\rho) = \rho \sqcup \sigma$ since $\emptyset^\ell \sim o$. Similarly $\sigma \sim \sigma \sqcup \rho$ as $\rho\backslash\sigma = \emptyset^k$. We get $\rho \sim \sigma$ by transitivity and applying (i). Now to prove iib, if we have $\tau\backslash\rho = \tau\backslash\sigma$ for all τ, we get $\emptyset^k = \rho\backslash\rho = \rho\backslash\sigma$. Similarly for $\sigma\backslash\rho = \emptyset^\ell$. Therefore $\rho \sim \sigma$ by (iii). To prove (iv), the right-to-left direction is implied by the definition of $\sim$. Now let $\rho\sigma \sim \rho\tau$, we have $\rho\sigma\backslash\rho\tau = (\rho\sigma\backslash\rho)\backslash\tau$. But $\rho\sigma\backslash\rho = (\rho\backslash\rho)(\sigma\backslash(\rho\backslash\rho)) = \emptyset^k(\sigma\backslash(\emptyset^k)) = \emptyset^k\sigma$. Therefore $\rho\sigma\backslash\rho\tau = (\emptyset^k\sigma)\backslash\tau = \emptyset^k(\sigma\backslash\tau)$. Hence $\rho\sigma\backslash\rho\tau \sim \sigma\backslash\tau$. Therefore by (iii), if $\rho\sigma \sim \rho\tau$, we also have $\sigma \sim \tau$. Q.E.D.

The permutation equivalence has many properties, most of them are due to the nice symmetry of the cube lemma. An interesting fact is that an upper semi-lattice can be constructed (or a push-out in categorical terminology). This lattice of reductions is derived by the following pre-ordering on coinitial reductions $\rho \leq \sigma$ iff $\rho\tau \sim \sigma$ for some τ. The interested reader is refer to [16, 3, 4, 11]. A permutation class of reductions is also characterized by its *unique* standard reduction. The permutation equivalence can also be viewed as the analogous of parse trees in context-free languages where several derivations equivalent by permutations correspond to the same parse tree. In formal grammars, the notion of ambiguity reveals two different parse trees corresponding to a same terminal sentence. In the lambda-calculus, this ambiguity often arises.

Take for instance $\rho : \Delta\Delta \to \Delta\Delta$ and $\sigma : \Delta\Delta \to \Delta\Delta \to \Delta\Delta$. Then ρ and σ are coinitial and cofinal, but $\rho \not\sim \sigma$. Similarly $o \not\sim \rho$. A more complex example is when the initial term M is $I((\lambda y.Ix)z)$. Then the reduction graph of M has no lattice structure, but the corresponding ordered structure induced by the permutation equivalence is indeed a lattice. Furthermore, the term M contains three redexes $R = I((\lambda y.Ix)z))$, $S = Ix$ and $K = (\lambda y.Ix)z$. When ρ and σ are two reductions defined by $\rho : I((\lambda y.Ix)z) \to (\lambda y.Ix)z \to Ix$ and $\sigma : I((\lambda y.Ix)z) \to I((\lambda y.x)z) \to Ix$, the hredex $\langle\rho, Ix\rangle$ is a residual of $\langle o, S\rangle$, but $\langle\sigma, Ix\rangle$ is a residual of $\langle o, R\rangle$. This kind of ambiguity cannot happen in the lattice of reductions where residuals are consistent with history, thanks to the cube lemma.



The interconnection between two created hredexes $\langle \rho, R\rangle$ and $\langle \sigma, S\rangle$ is not obvious as demonstrated by the following example. Take $M = \Delta(IIx)$ which contains two redexes $\Delta(IIx)$ and II. When the latter is contracted, it creates the new redex Ix which may appear in many reducts of M. One of them seems more representative, namely the one in $\Delta(Ix)$ where his history $\rho_0 : M \to \Delta(Ix)$ is entirely devoted to create the Ix redex. But to connect other instances of Ix in the reduction graph of M, it is necessary to close downward with residuals. For instance if $\rho_1 : M \to (IIx)(IIx) \to (Ix)(IIx)$ and $\rho_1 : M \to (IIx)(IIx) \to (IIx)(Ix)$, in order to connect $\langle \rho_1, Ix\rangle$ and $\langle \rho_2, Ix\rangle$, one has to link them to $\langle \rho_0, Ix\rangle$ through $\langle \rho_3, R_1\rangle$ and $\langle \rho_3, R_2\rangle$ where $\rho_3 : M \to (IIx)(IIx) \to (Ix)(IIx) \to (Ix)(Ix)$ (R_1 is left redex in $(Ix)(Ix)$ and R_2 is right redex in $(Ix)(Ix)$). More precisely $\langle \rho_1, Ix\rangle$ has residual $\langle \rho_3, R_1\rangle$ which is residual of $\langle \rho_0, Ix\rangle$ which has $\langle \rho_3, R_2\rangle$ as residual and this last one is residual of $\langle \rho_2, Ix\rangle$. Therefore created redexes can be connected by a zigzag of residuals related by consistent histories (i.e. using the permutation equivalence).



We can now define formally the residual and family relations between hredexes.

Definition 3. *Let two reductions* ρ *and* σ *be coinitial. The hredex* $\langle\sigma, S\rangle$ *is a residual of the hredex* $\langle\rho, R\rangle$ *iff there is a reduction* τ *such that* $\rho\tau \sim \sigma$ *and* $S \in R\backslash\tau$*. We write then* $\langle\rho, R\rangle \leqslant \langle\sigma, S\rangle$*.*

We say that the hredexes $\langle\rho, R\rangle$ *and* $\langle\sigma, S\rangle$ *are in the same family if they are connected by the reflexive, symmetric and transitive closure of the previous residual relation on hredexes. We then write* $\langle\rho, R\rangle \approx \langle\sigma, S\rangle$*. Formally:*

(i) $\langle\rho, R\rangle \approx \langle\rho, R\rangle$
(ii) $\langle\rho, R\rangle \approx \langle\sigma, S\rangle$ *implies* $\langle\sigma, S\rangle \approx \langle\rho, R\rangle$
(iii) $\langle\rho, R\rangle \approx \langle\sigma, S\rangle \approx \langle\tau, T\rangle$ *implies* $\langle\rho, R\rangle \approx \langle\tau, T\rangle$
(iv) $\langle\rho, R\rangle \leqslant \langle\sigma, S\rangle$ *implies* $\langle\rho, R\rangle \approx \langle\sigma, S\rangle$

We notice that the residual and family relations form respectively an ordering and an equivalence. The notation for residuals of hredexes is consistent since when $\rho\tau \sim \sigma \sim \rho\tau'$, we derive $\tau \sim \tau'$ giving $R\backslash\tau = R\backslash\tau'$. Therefore we may omit the specific reduction τ in the definition of residuals of hredexes. We can also see that the family relation is consistent with permutation by equivalences on reductions, since when $\rho \sim \sigma$, one has $\langle\rho, R\rangle \leqslant \langle\sigma, R\rangle$ since $\rho o = \rho \sim \sigma$ and $R \in R\backslash o$ and therefore $\langle\rho, R\rangle \approx \langle\sigma, R\rangle$. Again, many properties can be shown on redex families, one of the most striking is the connection with the labeled lambda-calculus. The interested reader is referred to [16]. We just mention here that when $\langle\rho, R\rangle$ and $\langle\sigma, S\rangle$ have a common residual $\langle\tau, T\rangle$, then there is T_0 such that the hredex $\langle\rho \sqcup \sigma, T_0\rangle$ is a residual of $\langle\rho, R\rangle$ and $\langle\sigma, S\rangle$. An alternative way is to state that $\langle\rho, R\rangle \leqslant \langle\tau, T\rangle$ and $\langle\sigma, S\rangle \leqslant \langle\tau, T\rangle$ implies there exists a (unique) $\langle\rho \sqcup \sigma, T_0\rangle$ such that $\langle\rho, R\rangle \leqslant \langle\rho \sqcup \sigma, T_0\rangle \leqslant \langle\tau, T\rangle$ and $\langle\sigma, S\rangle \leqslant \langle\rho \sqcup \sigma, T_0\rangle \leqslant \langle\tau, T\rangle$.

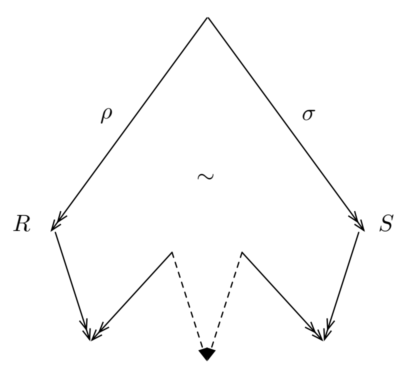


Fig. 3. $\langle\rho, R\rangle \approx \langle\sigma, S\rangle$: the hredexes $\langle\rho, R\rangle$ and $\langle\sigma, S\rangle$ belongs to the same family.

Now we consider the permutation equivalence and the family relation in the *hw*(L)-lambda-calculus.

Lemma 5. *Let ρ and σ be two reductions starting at M. Let U be a any term such that $||U|| = M$. Let L be sufficiently large such that ρ and σ can be performed in $hw(L)$. Then $\rho \sim \sigma$ implies that ρ and σ are cofinal in $hw(L)$.*

The proof is easy since the parallel-moves lemma can be applied in the $hw(L)$-lambda-calculus, since residuals keep redex degrees. Therefore, since $\rho \sim \sigma$, we have $\sigma \backslash \rho = \emptyset^n$ and $\rho \backslash \sigma = \emptyset^m$, which means ρ and σ are also cofinal in the $hw(L)$-lambda-calculus. Q.E.D.

For the interested reader, the labeled lambda-calculus in [16] characterizes exactly the permutation equivalence, i.e. two reductions are equivalent iff they are coinitial and cofinal in the labeled lambda-calculus.

An easy corollary of this lemma is that residuals of hredexes are the same notion in the standard lambda-calculus and in the $hw(\infty)$-calculus. Therefore the notion of redex families also coincide. A last remark is that if two redexes R and S are in the same family, their degree are the same (since residuals keep degrees).

5 Generalized finite developments

The usual finite development theorem deals with redexes who are in the initial term. Now created redexes can be taken into account with redex families.

Theorem 3 (GFD). *Let $\mathcal{F}$ be a finite set of redex families in $\mathcal{R}(M)$, consider relative reductions which contract only redexes belonging to a family in $\mathcal{F}$. Then:*

1. *there is no infinite reduction relative to $\mathcal{F}$;*
2. *the developments of $\mathcal{F}$ (maximal relative reductions) end all at the same term;*
3. *if ρ and σ are two developments of $\mathcal{F}$, then $\rho \sim \sigma$.*

The proof is rather straightforward since the family relation is identical in the usual lambda-calculus and in the $hw(\infty)$-lambda-calculus. Let U_0 be the term corresponding to M with all subterms with null exponents ($||U_0|| = M$). Redexes in each family have same degree. Let L be the largest degree for redexes in the finite family set $\mathcal{F}$. Then a reduction relative to $\mathcal{F}$ is inside the $hw(L)$-lambda-calculus. As $hw(L)$ is a canonical system, we know that these relative reductions cannot be infinite. Therefore all reductions relative to $\mathcal{F}$ are finite.

Now, if two reductions $\rho : M \twoheadrightarrow N$ and $\sigma : M \twoheadrightarrow P$ starting at M are complete developments of $\mathcal{F}$ (i.e. maximum relative reductions to $\mathcal{F}$), we notice that $\rho \sqcup \sigma$ and $\sigma \sqcup \rho$ are also reductions relative to $\mathcal{F}$ by definition of the family relation. As ρ and σ are maximal, we have $\sigma \backslash \rho = \emptyset^n$ and $\rho \backslash \sigma = \emptyset^m$. Therefore $\rho \sim \tau$. Hence we get $N = P$. But $\rho \sim \tau$ also means that if τ is another reduction starting at M, we have $\tau \backslash \rho = \tau \backslash \sigma$. Q.E.D.

Thus, the FD theorem has been extended to finite set of families. All developments of such a given finite set of redex families are equivalent by permutations, and therefore the consistency of residuals w.r.t. developments holds for the usual notion of redex residual and for residuals of reductions. It means that if a term M can start an infinite reduction, this infinite reduction contracts an infinite set of redex families. Hence a term can have an infinite reduction iff it can generates infinitely new redex families. This can be formalized in the following way.

Definition 4. *We say that the hredex $\langle\rho, R\rangle$ directly creates the hredex $\langle\sigma, S\rangle$ iff $\sigma \sim \rho R$ and the contraction of R creates S. We write $\langle\rho, R\rangle \blacktriangleleft \langle\sigma, S\rangle$.*

The hredex $\langle\rho, R\rangle$ creates $\langle\sigma, S\rangle$ if they are connected by transitive closure of direct creation and residual. We then write $\langle\rho, R\rangle \lhd \langle\sigma, S\rangle$. Formally:

(i) $\langle\rho, R\rangle \blacktriangleleft \langle\sigma, S\rangle$ *implies* $\langle\rho, R\rangle \lhd \langle\sigma, S\rangle$
(ii) $\langle\rho, R\rangle \leqslant \langle\rho', R'\rangle \lhd \langle\sigma, S\rangle$ *implies* $\langle\rho, R\rangle \lhd \langle\sigma, S\rangle$
(iii) $\langle\rho, R\rangle \lhd \langle\sigma', S'\rangle \leqslant \langle\sigma, S\rangle$ *implies* $\langle\rho, R\rangle \lhd \langle\sigma, S\rangle$
(iv) $\langle\rho, R\rangle \lhd \langle\sigma, S\rangle \lhd \langle\tau, T\rangle$ *implies* $\langle\rho, R\rangle \lhd \langle\tau, T\rangle$

So $\langle\rho, R\rangle$ directly creates $\langle\sigma, S\rangle$ if the reduction σ is equivalent by permutations to a reduction $M \twoheadrightarrow N \xrightarrow{R} P$ where the first part $M \twoheadrightarrow N$ is the reduction ρ and the contraction of R creates the new redex S in P, i.e. $S \notin S'\backslash R$ for any S' in N. Therefore the notion of creation is consistent with the equivalence $\sim$. One also easily checks that the creation relation is antisymmetric, since with the $hw(\infty)$-lambda-calculus we know that degrees of redexes are kept invariant by residuals and increase in case of a direct creation. So the creation relation $\lhd$ is a strict ordering. We would like to prove that a term which does not contain an infinite chain of creations is strongly normalizable. Unfortunately this property seems to need more advanced technical arguments. We will show a broader property relying on the creation of families.

Definition 5. *The family of the hredex $\langle\rho, R\rangle$ creates the family of the hredex $\langle\sigma, S\rangle$ if a hredex in the family of $\langle\rho, R\rangle$ creates a hredex in the family of $\langle\sigma, S\rangle$. We write $\langle\rho, R\rangle \ll \langle\sigma, S\rangle$. Formally:*

(i) $\langle\rho, R\rangle \blacktriangleleft \langle\sigma, S\rangle$ *implies* $\langle\rho, R\rangle \ll \langle\sigma, S\rangle$
(ii) $\langle\rho, R\rangle \approx \langle\rho', R'\rangle \ll \langle\sigma, S\rangle$ *implies* $\langle\rho, R\rangle \ll \langle\sigma, S\rangle$
(iii) $\langle\rho, R\rangle \ll \langle\sigma', S'\rangle \approx \langle\sigma, S\rangle$ *implies* $\langle\rho, R\rangle \ll \langle\sigma, S\rangle$
(iv) $\langle\rho, R\rangle \ll \langle\sigma, S\rangle \ll \langle\tau, T\rangle$ *implies* $\langle\rho, R\rangle \ll \langle\tau, T\rangle$

Clearly this relation with creation of families is broader than the previous relation on creation of hredexes, since $\langle\rho, R\rangle \approx \langle\sigma, S\rangle$ when $\langle\rho, R\rangle \leqslant \langle\sigma, S\rangle$. It also defines a strict ordering on families with the same argument as the one used for hredex creation. With this new ordering, one can easily show that strong normalization is induced by the absence of infinite chains of family creations.

Lemma 6. *A term without infinite chains of family creations is strongly normalizable.*

We consider the covering relation of the $\ll$ ordering (i.e. $\langle\rho, R\rangle \ll \langle\sigma, S\rangle$ when there is no intermediate $\langle\tau, T\rangle$ between $\langle\rho, R\rangle$ and $\langle\sigma, S\rangle$). This relation induces a directed acyclic graph with finite number of successors for each node. To see that the out-degree of each node is finite, one can work by induction on the length from the roots of this graph since for a given length n we know by the gFD theorem that we have the strong normalization property and therefore a finite number of redex families. Now if the total number of families (i.e. number of nodes in the dag) is infinite, we get by Koenig lemma that there is an infinite path from the roots. It means that there is an infinite chain of family creations, which is impossible by hypothesis of the lemma. So the total number of families for the given term is finite. By the gFD theorem, we get that this term strongly normalizes. Q.E.D.

As corollary, we get strong normalization for the first-order typed lambda-calculus. In this calculus, we call degree of a redex the type of its function part. We notice that a redex of degree $\alpha \to \beta$ can only creates redexes of type α or β (this can be proved by case inspection as in the paragraph following lemma 1). Furthermore, when two redexes are in the same family, their degrees are equal. Thus, there are no infinite chains of family creations for any term of the first-order typed lambda-calculus. By previous lemma, the calculus enjoys strong normalization.

6 Conclusion

The generalized Finite Development (gFD) theorem shows that relative reductions w.r.t. a finite set of redex families cannot be infinite and that all developments end with the same term.

The gFD theorem is an important tool in the understanding of strong normalisation. It defines arbitrary large canonical systems as subgraphs of the reduction graph of any term. In this sense, it expresses the *compactness* of the lambda-calculus. It was used in [15] for proving the completeness of inside-out reductions, i.e. for any ρ there is an inside-out reduction σ and a (inside-out) reduction τ such that $\rho\tau \sim \sigma$. But we can get completeness of many more reduction strategies with the gFD theorem as long as the strategy consists in reordering of contractions of redexes in the same family set.

Furthermore, it is an argument for proving strong normalization as soon as one can prove that there is no infinite chain of family creations. With the gFD theorem, the proof of strong normalization is reduced to the proof of the non existence of such infinite chains of creations. We took the example of the standard first-order typed lambda-calculus, but some effort remains to be done in the understanding of higher-order typed systems where we conjecture that such an argument must exist. If correct, we could get illuminating proofs of strong normalization.

The gFD theorem also holds in other calculi. In fact, this theorem is true in any calculus with some compactness property, i.e. nearly all calculi. In the lambda-calculus, it is simple to express, but it can also be easily stated in orthogonal term rewriting systems [11] or in combinatory reduction systems [12]. This theorem seems also correct in calculi with critical pairs, where permutation equivalence is more complex to define (see [5]) since the equivalence classes may be incompatibles due to different choices around critical pairs. Then the permutation equivalence is very close to event structures [20, 14].

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