Concurrency 4

Bisimulations

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CCS with values (1/4)

Language

$$
\begin{array}{rcll}
x & ::= & \text{variables} & \\
\tilde{x} & ::= & x_1, x_2, \ldots x_n \quad (n \geq 0) & \\
v & ::= & \text{values} & \\
\tilde{v} & ::= & v_1, v_2, \ldots v_n \quad (n \geq 0) & \\
a, b, c & ::= & \text{(channel) names} & \\
\overline{a}, \overline{b}, \overline{c} & ::= & \text{co-names} \qquad \overline{\overline{a}} = a & \\
\alpha & ::= & a(x) \mid \overline{a}v \mid \tau & \text{actions} \\
P, Q, R & ::= & 0 \mid \alpha.P \mid P + Q \mid (P \mid Q) \mid (\nu\alpha)P \mid K\langle\tilde{v}\rangle & \text{processes} \\
K\langle\tilde{x}\rangle \overset{\text{def}}{=} P & ::= & \text{constant definitions} &
\end{array}
$$

$$\mathcal{A}ct = \{a(x), b(x), c(x), \ldots\} \cup \{\overline{a}v, \overline{b}v, \overline{c}v, \ldots\} \cup \{\tau\}$$

Notation: α for $\alpha.0$

CCS with values (2/4)

Memory register

$$\begin{aligned} Reg\langle\rangle &\stackrel{\text{def}}{=} \mathsf{put}(x).A\langle x\rangle \\ A\langle x\rangle &\stackrel{\text{def}}{=} \mathsf{put}(y).A\langle y\rangle \mid \overline{\mathsf{get}}\,x.A\langle x\rangle \end{aligned}$$

$$\ldots \mid P \mid \overline{\mathsf{put}}\,1 \mid \mathsf{get}(x).Q \mid \overline{\mathsf{put}}\,2.\,\mathsf{get}(y).R \mid \ldots$$

Exercice 1 What can be values of x and y in Q and R ?

Buffers

$$Buf_1{}^{\mathsf{in},\mathsf{out}}\langle\rangle \stackrel{\text{def}}{=} \mathsf{in}(x).\overline{\mathsf{out}}\,x.\,Buf_1{}^{\mathsf{in},\mathsf{out}}\langle\rangle$$

$$\begin{aligned} Buf_2{}^{\mathsf{in},\mathsf{out}}\langle\rangle &\stackrel{\text{def}}{=} \mathsf{in}(x).A\langle x\rangle \\ A\langle x\rangle &\stackrel{\text{def}}{=} \mathsf{in}(y).\overline{\mathsf{out}}\,x.A\langle y\rangle + \overline{\mathsf{out}}\,x.\,Buf_2{}^{\mathsf{in},\mathsf{out}}\langle\rangle \end{aligned}$$

$$Buf_1{}'\langle\rangle \stackrel{\text{def}}{=} (\nu c)(Buf_1{}^{\mathsf{in},c}\langle\rangle \mid Buf_1{}^{c,\mathsf{out}}\langle\rangle)$$

Exercice 2 Relate Buf_2 and $Buf_1{}'$.

CCS with values (3/4)

Semantics (SOS)

[Act] $\alpha.P \xrightarrow{\alpha} P$

[Sum1] $\dfrac{P \xrightarrow{\alpha} P'}{P + Q \xrightarrow{\alpha} P'}$

[Sum2] $\dfrac{Q \xrightarrow{\alpha} Q'}{P + Q \xrightarrow{\alpha} Q'}$

[Par1] $\dfrac{P \xrightarrow{\alpha} P'}{P \mid Q \xrightarrow{\alpha} P' \mid Q}$

[Par2] $\dfrac{Q \xrightarrow{\alpha} Q'}{P \mid Q \xrightarrow{\alpha} P \mid Q'}$

[Com1] $\dfrac{P \xrightarrow{a(x)} P' \quad Q \xrightarrow{\overline{a}v} Q'}{P \mid Q \xrightarrow{\tau} P'\{v/x\} \mid Q'}$

[Com2] $\dfrac{P \xrightarrow{\overline{a}v} P' \quad Q \xrightarrow{a(x)} Q'}{P \mid Q \xrightarrow{\tau} P' \mid Q'\{v/x\}}$

[Res] $\dfrac{P \xrightarrow{\alpha} P' \quad \alpha \notin \{a, \overline{a}\}}{(\nu a)P \xrightarrow{\alpha} (\nu a)P'}$

[Rec] $\dfrac{P\{\tilde{v}/\tilde{x}\} \xrightarrow{\alpha} P' \quad K\langle\tilde{x}\rangle \stackrel{\text{def}}{=} P}{K\langle\tilde{v}\rangle \xrightarrow{\alpha} P'}$

CCS with values (4/4)

One can emulate CCS with values by pure CCS (with infinite sum).

P	$[\![P]\!]$	
$a(x).P$	$\Sigma_{v \in \mathcal{V}} a_v.[\![P\{v/x\}]\!]$	($\mathcal{V}$ set of values)
$\overline{a}v.P$	$a_v.[\![P]\!]$	
$\tau.P$	$\tau.[\![P]\!]$	
$\Sigma_{i \in I} P_i$	$\Sigma_{i \in I} [\![P_i]\!]$	
$P \mid Q$	$[\![P]\!] \mid [\![Q]\!]$	
$\ldots$		

Exercice 3 Terminate the translation

Exercice 4 Find the relation between P and $[\![P]\!]$.

CCS and weak bisimulation (1/4)

Write $P(\xrightarrow{\tau})^*Q$ if $P = P_0 \xrightarrow{\tau} P_1 \xrightarrow{\tau} P_2 \cdots \xrightarrow{\tau} P_k = Q \qquad (k \geq 0)$.

Let $\mu = \mu_1\mu_2\cdots\mu_n \qquad (n > 0)$

Write $P \xrightarrow{\mu} Q$ if $P \xrightarrow{\mu_1}\xrightarrow{\mu_2} \cdots \xrightarrow{\mu_n} Q$

Write $P \stackrel{\mu}{\Longrightarrow} Q$ if $P(\xrightarrow{\tau})^* \xrightarrow{\mu_1} (\xrightarrow{\tau})^* \xrightarrow{\mu_2} (\xrightarrow{\tau})^* \cdots \xrightarrow{\mu_n} (\xrightarrow{\tau})^* Q$

Write $\widehat{\mu}$ be μ where all occurences of τ in μ have been erased.

Take $\mu = \tau a b \tau\tau\overline{a}$, then $\widehat{\mu} = ab\overline{a}$. If $\mu = \tau^n$, then $\widehat{\mu} = \epsilon$ (empty string).

Then $\xrightarrow{\mu}$ specifies exactly the τ actions occuring in μ

$\stackrel{\mu}{\Longrightarrow}$ specifies at least the τ actions

$\stackrel{\widehat{\mu}}{\Longrightarrow}$ specifies nothing about the τ actions

CCS and weak bisimulation (2/4)

Definition 1 P weakly bisimilar to Q (we write $P \approx Q$) if, for any $\alpha \in \mathcal{A}ct$, whenever

- $P \xrightarrow{\alpha} P'$, there is Q' such that $Q \stackrel{\widehat{\alpha}}{\Longrightarrow} Q'$ and $P' \approx Q'$.
- $Q \xrightarrow{\alpha} Q'$, there is P' such that $P \stackrel{\widehat{\alpha}}{\Longrightarrow} P'$ and $P' \approx Q'$.

($\approx$ is the largest weak bisimulation)

Examples

$A \stackrel{\text{def}}{=} c.(k.A + t.A)$ $\qquad$ $C \stackrel{\text{def}}{=} (a.\overline{b}.\tau.C + \overline{b}.a.\tau.C$

$B \stackrel{\text{def}}{=} (\nu d)c.(k.d.B + t.d.B)$ $\qquad$ $D \stackrel{\text{def}}{=} ((a.\overline{b}.D + \overline{b}a.D)$

$A \approx B$ $\qquad$ $C \approx D$

Exercice 5 Find weak bisimulation when

$A_0 \stackrel{\text{def}}{=} a.A_0 + b.A_1 + \tau.A_1$

$A_1 \stackrel{\text{def}}{=} a.A_1 + \tau.A_2$ $\qquad$ $B_1 \stackrel{\text{def}}{=} a.B_1 + \tau.B_2$

$A_2 \stackrel{\text{def}}{=} b.A_0$ $\qquad$ $B_2 \stackrel{\text{def}}{=} b.B_1$

CCS and weak bisimulation (3/4)

Proposition 2 Following equations hold.

$$\begin{array}{rcl} \tau.P & \approx & P \\ P \approx Q & \Rightarrow & P \mid R \approx Q \mid R \\ P \approx Q & \Rightarrow & R \mid P \approx R \mid Q \\ P \approx Q & \Rightarrow & \alpha.P \approx \alpha.Q \\ P \approx Q & \Rightarrow & (\nu x)P \approx (\nu x)Q \\ K \overset{\text{def}}{=} P \text{ and } P \approx Q & \Rightarrow & K \approx Q \end{array}$$

Exercice 6 Prove it.

Fact 3 $P \approx Q \not\Rightarrow P + R \approx Q + R$

Take $P = \tau.Q$, $Q = a$, $R = b$.
Then $\tau.a \approx a$. But we have not $\tau.a + b \approx a + b$.

Hence weak bisimulation is not a congruence in CCS
(differs from strong bisimulation)

CCS and weak bisimulation (4/4)

Definition 4 [observation-congruence] P weakly congruent to Q (we write $P \cong Q$) if, for any $\alpha \in \mathcal{A}ct$, whenever

- $P \xrightarrow{\alpha} P'$, there is Q' such that $Q \stackrel{\alpha}{\Longrightarrow} Q'$ and $P' \approx Q'$.
- $Q \xrightarrow{\alpha} Q'$, there is P' such that $P \stackrel{\alpha}{\Longrightarrow} P'$ and $P' \approx Q'$.

Exercice 7 Prove $\cong \subseteq \approx$.

Proposition 5 $\cong$ is a congruence.

Exercice 8 Prove it.

Proposition 6 The following τ laws are true:

$$\begin{aligned} \alpha.\tau.P &\approx \alpha.P \\ P + \tau.P &\approx \tau.P \\ \alpha.(P+\tau.Q) + \alpha.Q &\approx \alpha.(P+\tau.Q) \end{aligned}$$

Exercice 9 Prove them.

CCS and weak bisimulation (4/4)

Proposition 7

$$P \sim Q \;\Rightarrow\; P \cong Q \;\Rightarrow\; P \approx Q$$
$$P \approx Q \;\Rightarrow\; \alpha.P \cong \alpha.Q$$
$$P \approx Q \;\text{ iff }\; P \cong Q \text{ or } P \cong \tau.Q \text{ or } \tau.P \cong Q$$

Exercice 10 Prove it.

Proposition 8 $\cong$ is the largest congruence on CCS contained in $\approx$.

Exercice 11 Prove it.