Reductions and Causality (V)

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http://pauillac.inria.fr/~levy/courses/tsinghua/reductions

Plan

- a labeled λ-calculus
- lattice of reductions
- labels and redex families
- strong normalization
- canonical representatives
- Hyland-Wadsworth labeled calculus
- labels and types

Labeled λ-calculus

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A labeled λ-calculus (1/2)

- Give names to every redex and try make this naming consistent with permutation equivalence.

- Need give names to every subterm:

$$M, N, \ldots ::= {}^{\alpha}x \mid {}^{\alpha}(MN) \mid {}^{\alpha}(\lambda x.M)$$

- Conversion rule is:

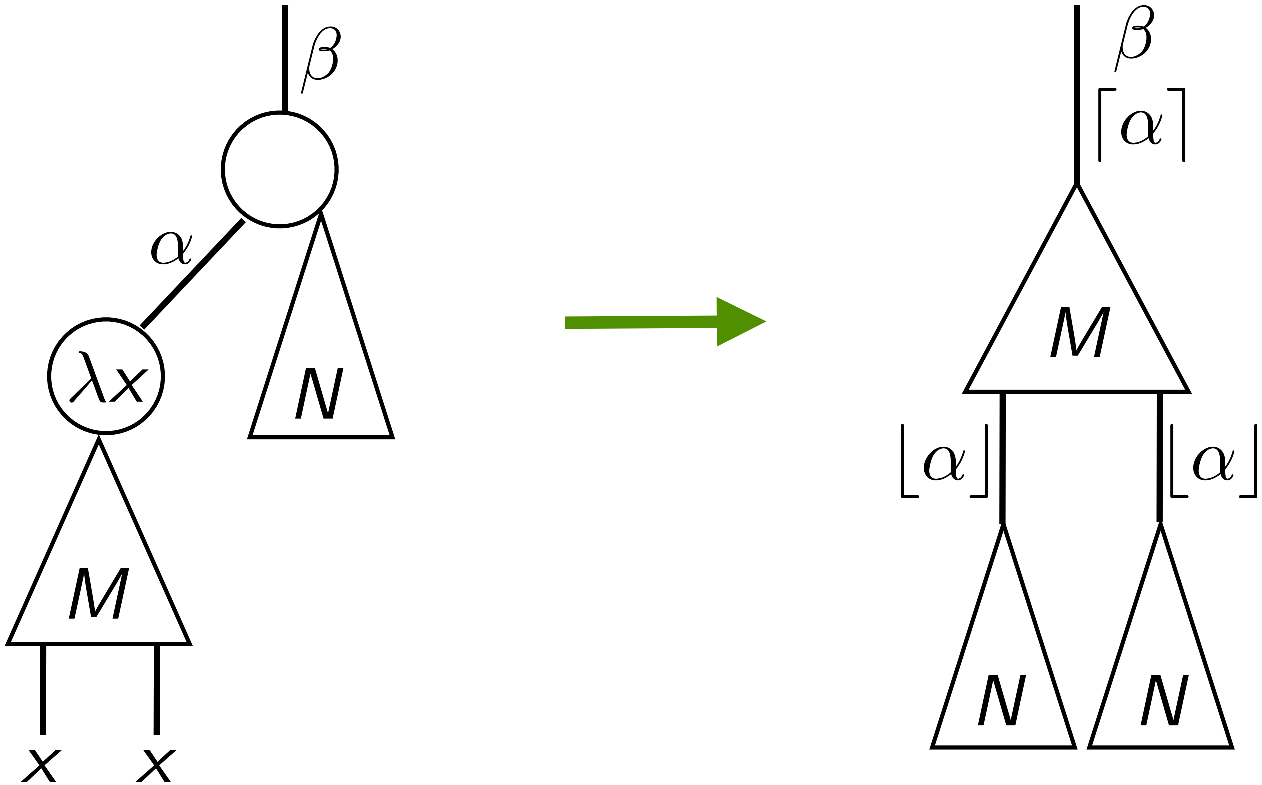
$${}^{\beta}({}^{\alpha}(\lambda x.M)N) \longrightarrow {}^{\beta\lceil\alpha\rceil}M\{x := {}^{\lfloor\alpha\rfloor}N\}$$

α is **name** of redex

where

$${}^{\alpha}({}^{\beta}U) = {}^{\alpha\beta}U \quad \text{and} \quad {}^{\alpha}x\{x := M\} = {}^{\alpha}M$$

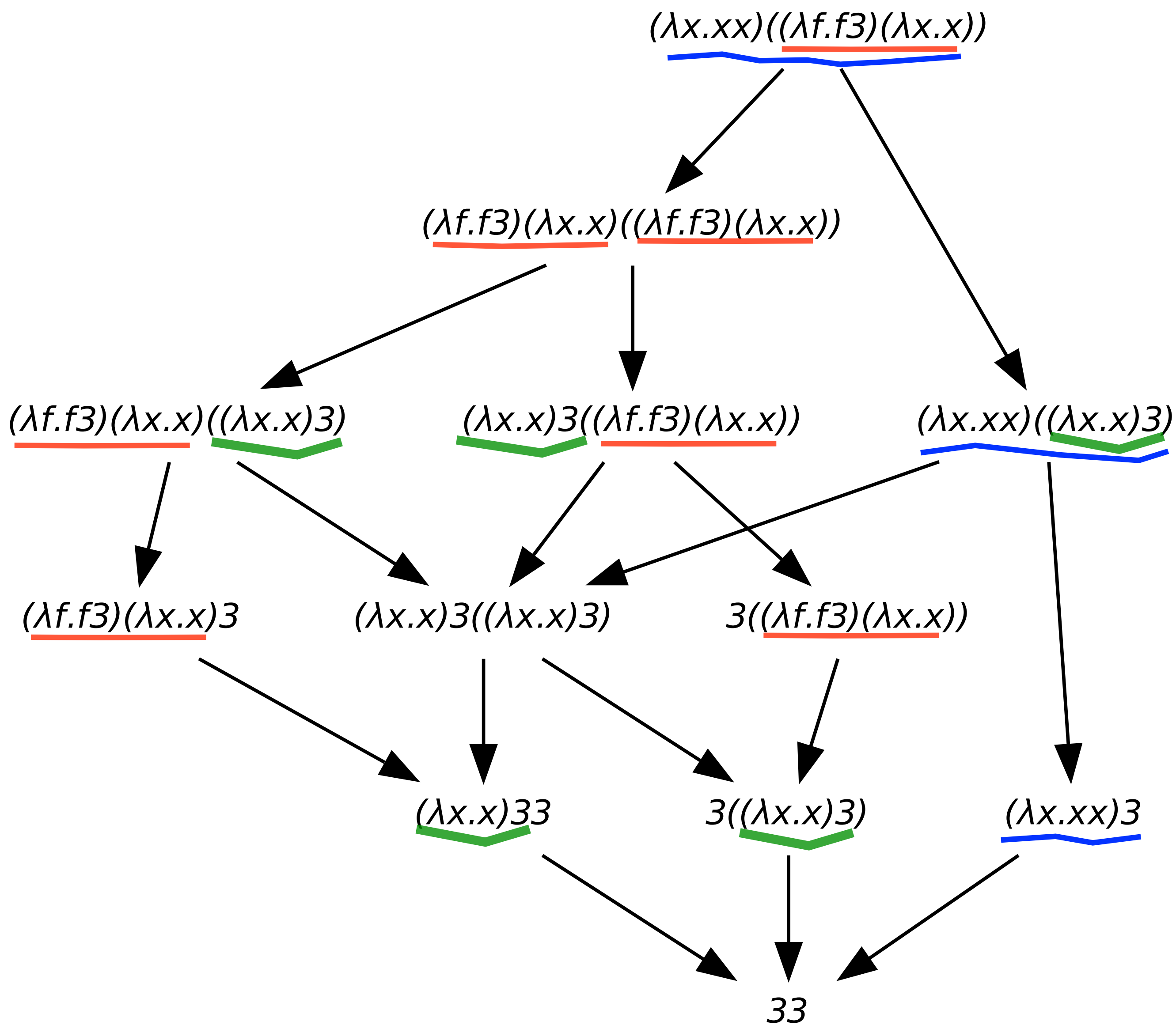
A labeled λ-calculus (2/2)



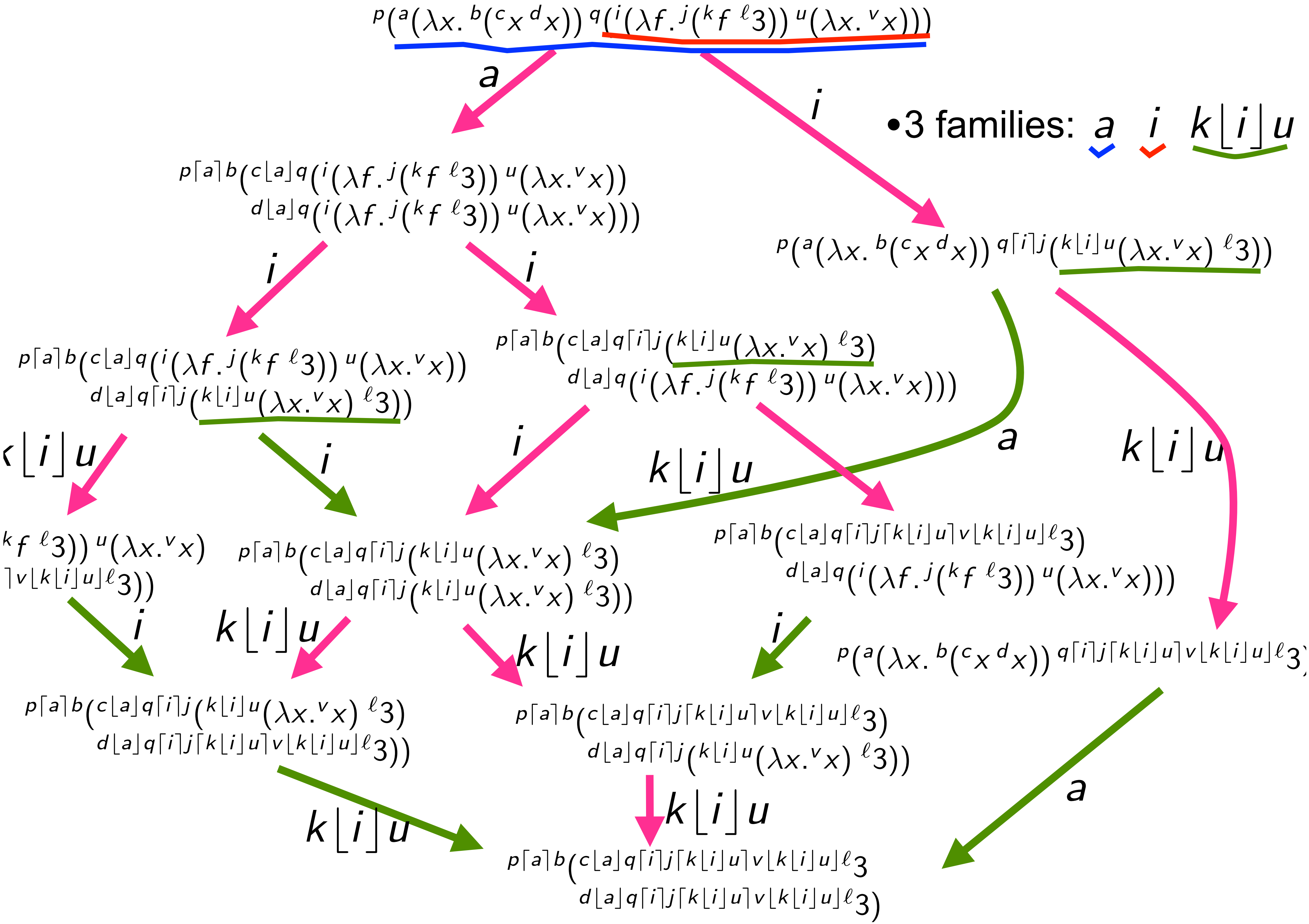
- Labels are nonempty strings of atomic labels:

$$\alpha, \beta, \ldots \quad ::= \quad \underbrace{a, b, c, \ldots \mid \lceil\alpha\rceil \mid \lfloor\alpha\rfloor}_{\texttt{atomic labels}} \mid \alpha\beta$$

Our favorite example



- 3 redex families: red, blue, green.


•3 families: a i $k\lfloor i\rfloor u$
${}^{p}({}^{a}(\lambda x.\,{}^{b}({}^{c}x\;{}^{d}x))\;{}^{q}({}^{i}(\lambda f.\,{}^{j}({}^{k}f\;{}^{\ell}3))\;{}^{u}(\lambda x.\,{}^{v}x)))$

Labels and permutation equiv. (1/7)

- **Proposition** [residuals of labeled redexes]

  $S \in R/\rho$ implies $\mathrm{name}(R) = \mathrm{name}(S)$

- **Definition** [created redexes] Let $\langle \rho, R \rangle$ be historical redex.

  We say that ρ **creates** R when $\nexists R', \ R \in R'/\rho$.

- **Proposition** [created labeled redexes]

  If S creates R, then $\mathrm{name}(S)$ is strictly contained in $\mathrm{name}(R)$.

Labels and permutation equiv. (2/7)

Proof (cont'd) Created redexes contains name of creator

$$\underbrace{{}^{\alpha}(\lambda x.\cdots({}^{\beta}x\,N)\cdots)}_{\alpha}\,{}^{\gamma}(\lambda y.M) \rightarrow \cdots(\underbrace{{}^{\beta\lfloor\alpha\rfloor\gamma}(\lambda y.M)N')}_{\beta\lfloor\alpha\rfloor\gamma}\cdots$$

creates

$$ {}^{\beta}(\underbrace{{}^{\alpha}(\lambda x.{}^{\gamma}(\lambda y.M)N)}_{\alpha}P \rightarrow (\underbrace{{}^{\beta\lceil\alpha\rceil\gamma}(\lambda y.M')P)}_{\beta\lceil\alpha\rceil\gamma}$$

creates

$$ {}^{\beta}(\underbrace{{}^{\alpha}(\lambda x.{}^{\gamma}x)\,{}^{\delta}(\lambda y.M))}_{\alpha}\,N \rightarrow \underbrace{{}^{\beta\lceil\alpha\rceil\gamma\lfloor\alpha\rfloor\delta}(\lambda y.M)N}_{\beta\lceil\alpha\rceil\gamma\lfloor\alpha\rfloor\delta}$$

creates

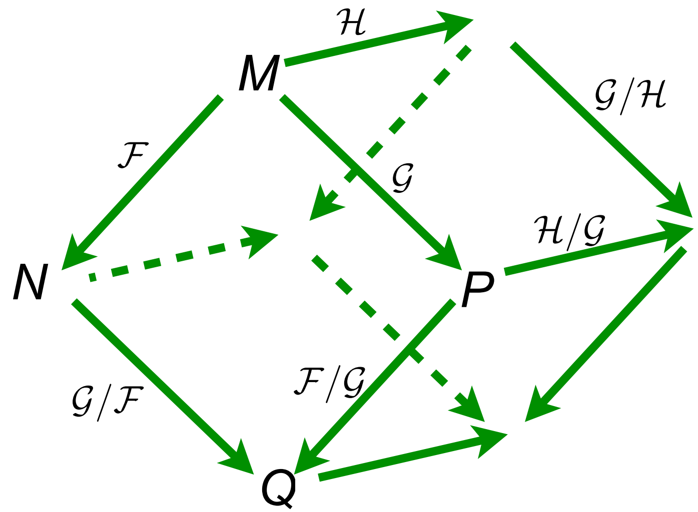
Labels and permutation equiv. (3/7)

- **Labeled parallel moves lemma+** [74]

If $M \xrightarrow{\mathcal{F}} N$ and $M \xrightarrow{\mathcal{G}} P$, then $N \xrightarrow{\mathcal{G}/\mathcal{F}} Q$ and $P \xrightarrow{\mathcal{F}/\mathcal{G}} Q$ for some Q.

- **Parallel moves lemma++** [The Cube Lemma] still holds.

$$(\mathcal{H}/\mathcal{F})/(\mathcal{G}/\mathcal{F}) = (\mathcal{H}/\mathcal{G})/(\mathcal{F}/\mathcal{G})$$



- **Finite developments** still holds.

Labels and permutation equiv. (4/7)

- Labels do not break Church-Rosser, nor residuals

- Labels refine λ-calculus:
  - any unlabeled reduction can be performed in the labeled calculus
  - but two cofinal unlabeled reductions may no longer be cofinal

  Take $I(I3)$ with $I = \lambda x.x$.

$$^{a}(^{b}(\lambda x.^{c}x)\ ^{d}(^{e}(\lambda x.^{f}x)\ ^{g}3))$$

$$^{a\lceil b\rceil c\lfloor b\rfloor d}(^{e}(\lambda x.^{f}x)\ ^{g}3) \qquad\qquad ^{a}(^{b}(\lambda x.^{c}x)\ ^{d\lceil e\rceil f\lfloor e\rfloor g}3)$$

Labels and permutation equiv. (5/7)

- **Theorem** `[labeled permutation equivalence, 76]`

  Let ρ and σ be coinitial reductions.
  Then $\rho \simeq \sigma$ iff ρ and σ are labeled cofinal.

  Proof Let $\rho \simeq \sigma$. Then obvious because of labeled parallel moves lemma.

  Conversely, we apply standardization thm and following lemma.

- **Lemma** `[uniqueness of labeled standard reductions]`

  Proof ...

Labels and permutation equiv. (6/7)

Proof [uniqueness of labeled standard

Let ρ and σ be 2 distinct coinitial standard reductions.

Take first step when they diverge. Call M that term.
We make structural induction on M. Say ρ is more to the left.
If first step of ρ contracts an internal redex, we use induction.
If first step of ρ contracts an external redex, then:

$$\begin{tikzcd}
 & M = {}^{\beta}({}^{\alpha}(\lambda x.P)Q) \arrow[dl] \arrow[ddr, bend left, "\mathsf{st}"] & \\
{}^{\beta\lceil\alpha\rceil}P\{x := {}^{\lfloor\alpha\rfloor}Q\} \arrow[d, "\mathsf{st}"] & & \\
{}^{\beta\lceil\alpha\rceil}N & \neq & {}^{\beta}({}^{\alpha}(\lambda x.A)B)
\end{tikzcd}$$

Labels and permutation equiv. (7/7)

- **Corollary** [labeled prefix ordering]

  Let $\rho : M \xrightarrow{\star} N$ and $\sigma : M \xrightarrow{\star} P$ be coinitial labeled reductions.
  Then $\rho \sqsubseteq \sigma$ iff $N \xrightarrow{\star} P$.

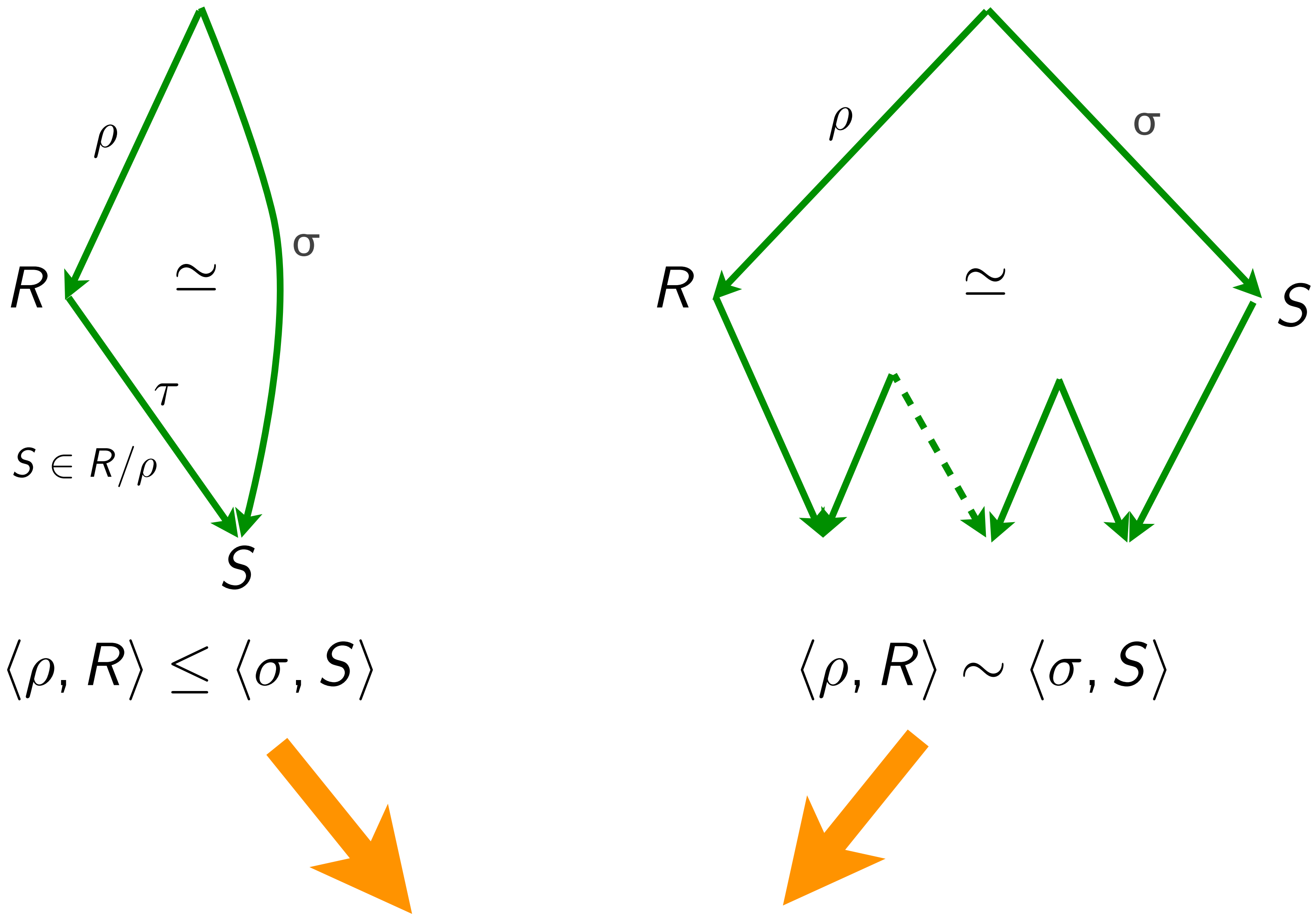
- **Corollary** [lattice of labeled reductions]

  Labeled reduction graphs are upwards semi lattices for any labeling.

- **Exercice** Try on $(\lambda x.x)((\lambda y.(\lambda x.x)a)b)$ or $(\lambda x.xx)(\lambda x.xx)$

Redex families

Labeled redexes and their history (1/3)



$\langle \rho, R \rangle \leq \langle \sigma, S \rangle$

$\langle \rho, R \rangle \sim \langle \sigma, S \rangle$

$\mathrm{name}(R) = \mathrm{name}(S)$

Labeled redexes and their history (2/3)

- **Proposition** `[same history → same name]`

  In the labeled λ-calculus, for any labeling, we have:

  $$\langle \rho, R \rangle \sim \langle \sigma, S \rangle \ \text{ implies } \ \mathsf{name}(R) = \mathsf{name}(S)$$

- The opposite direction is clearly not true for any labeling

  (For instance, take all labels equal)

- But it is true when all labels are distinct atomic letters in the initial term.

- **Definition** `[all labels distinct letters]`

  $\mathsf{INIT}(M) = \mathtt{True}$ when all labels in M are distinct letters.

Labeled redexes and their history (3/3)

- **Theorem** `[same history = same name, 76]`

  When $\mathrm{INIT}(M)$ and reductions ρ and σ start from M :

  $$\langle \rho, R \rangle \sim \langle \sigma, S \rangle \quad \text{iff} \quad \mathrm{name}(R) = \mathrm{name}(S)$$

- **Corollary** `[decidability of family relation]`

  The family relation is decidable (although complexity is proportional to length of standard reduction).

Strong normalization

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Strong normalization (1/3)

- Another labeled λ-calculus was considered to study Scott D-infinity model [Hyland-Wadsworth, 74]

- D-infinity projection functions on each subterm (*n* is any integer):

$$M, N, \ldots ::= x^n \mid (MN)^n \mid (\lambda x.M)^n$$

- Conversion rule is:

$$((\lambda x.M)^{n+1} N)^p \longrightarrow M\{x := N_{[n]}\}_{[n][p]}$$

$n+1$ is **degree** of redex

$$U_{[m][n]} = U_{[p]} \quad \text{where} \quad p = \min\{m, n\}$$

$$x^n\{x := M\} = M_{[n]}$$

Strong normalization (2/3)

- **Proposition** Hyland-Wadsworth calculus is derivable from labeled calculus by simple homomorphism on labels.

  Proof Assign an integer to any atomic letter and take:

  $$\mathsf{h}(\alpha\beta) = \mathsf{min}\{\mathsf{h}(\alpha),\, \mathsf{h}(\beta)\}$$

  $$\mathsf{h}(\lceil\alpha\rceil) = \mathsf{h}(\lfloor\alpha\rfloor) = \mathsf{h}(\alpha) - 1$$

- **Proposition** Hyland-Wadsworth calculus strongly normalizes.

- **Corollary** When only a finite set of redex names is contracted, there is strong normalization.