Reductions and Causality (IV)

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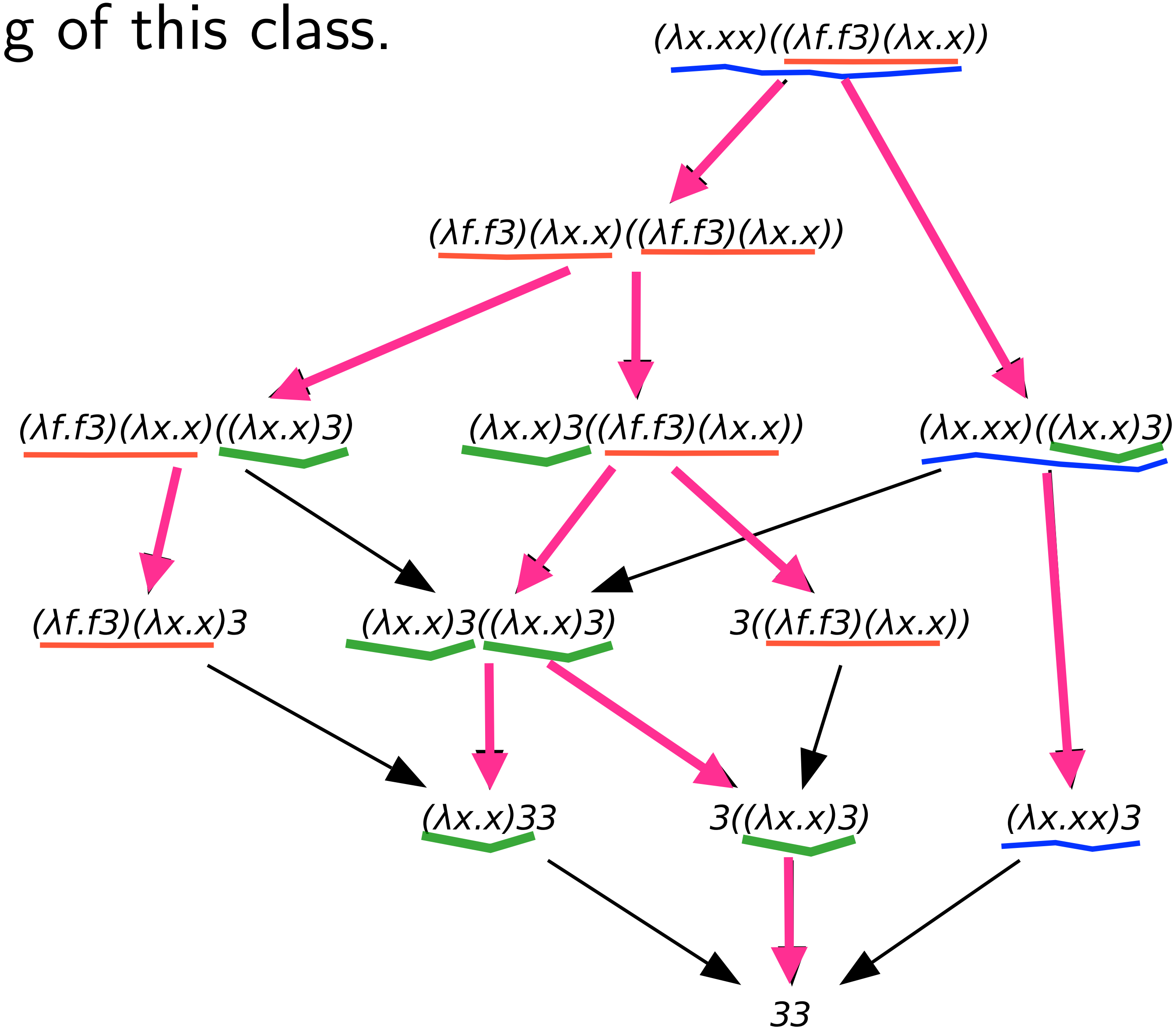
http://pauillac.inria.fr/~levy/courses/tsinghua/reductions

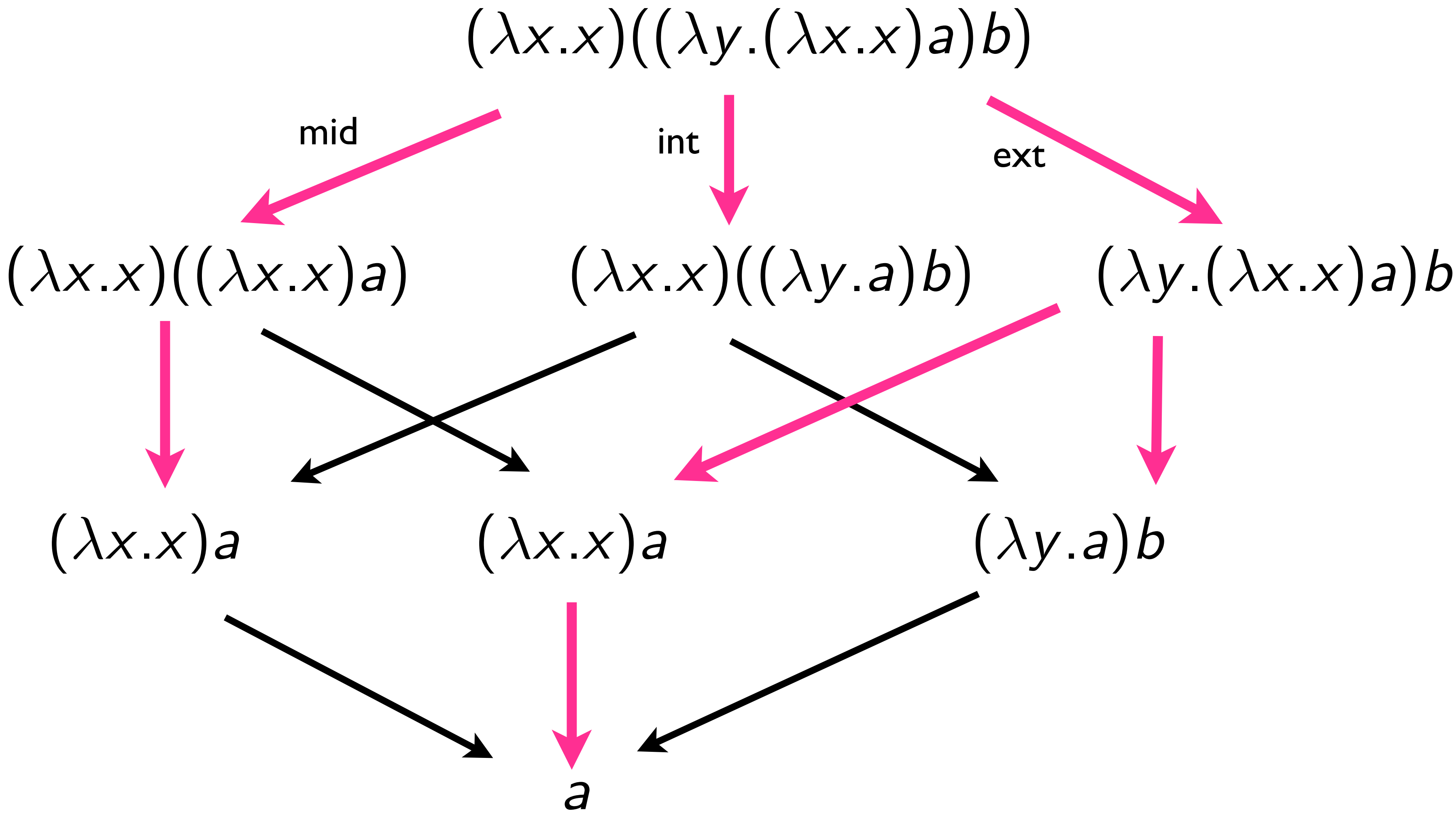
Exercices

- Show all standard reductions in the 2 reduction graphs of beginning of this class.



Exercices

- Show all standard reductions in the 2 reduction graphs of beginning of this class.



Exercices

- Find an example where there is no greatest lower bound of 2 reductions. (Hint: you should use K-terms)

$$\begin{tikzcd}
 & & M = (\lambda x.(\lambda y.a)(xY_f))(\lambda z.b) \arrow[ddll] \arrow[d] \arrow[dddrr] & & \\
 & & (\lambda x.(\lambda y.a)(xY_f^n))(\lambda z.b) \arrow[d] \arrow[d, dashed, shift left=2] \arrow[ddrr] & & \\
(\lambda y.a)((\lambda z.b)Y_f) \arrow[d] & & (\lambda y.a)((\lambda z.b)Y_f^n) \arrow[dll] \arrow[drr] & & \\
(\lambda y.a)b & & & & (\lambda x.a)(\lambda z.b)
\end{tikzcd}$$

$$Y_f = (\lambda x.f(xx))(\lambda x.f(xx))$$

Exercices

- Show that there is inf-lattice of reductions in λI-calculus.

  $\rho_{\mathsf{st}} : M \xrightarrow{\star} N,\ \sigma_{\mathsf{st}} : M' \xrightarrow{\star} N,\ \tau : M \xrightarrow{\star} M'$

  then $|\rho_{\mathsf{st}}| \geq |\sigma_{\mathsf{st}}| + |\tau|$

Plan

- redexes and their history
- creation of redexes
- redex families
- finite developments
- finite developments+
- infinite reductions, strong normalization

Redex families

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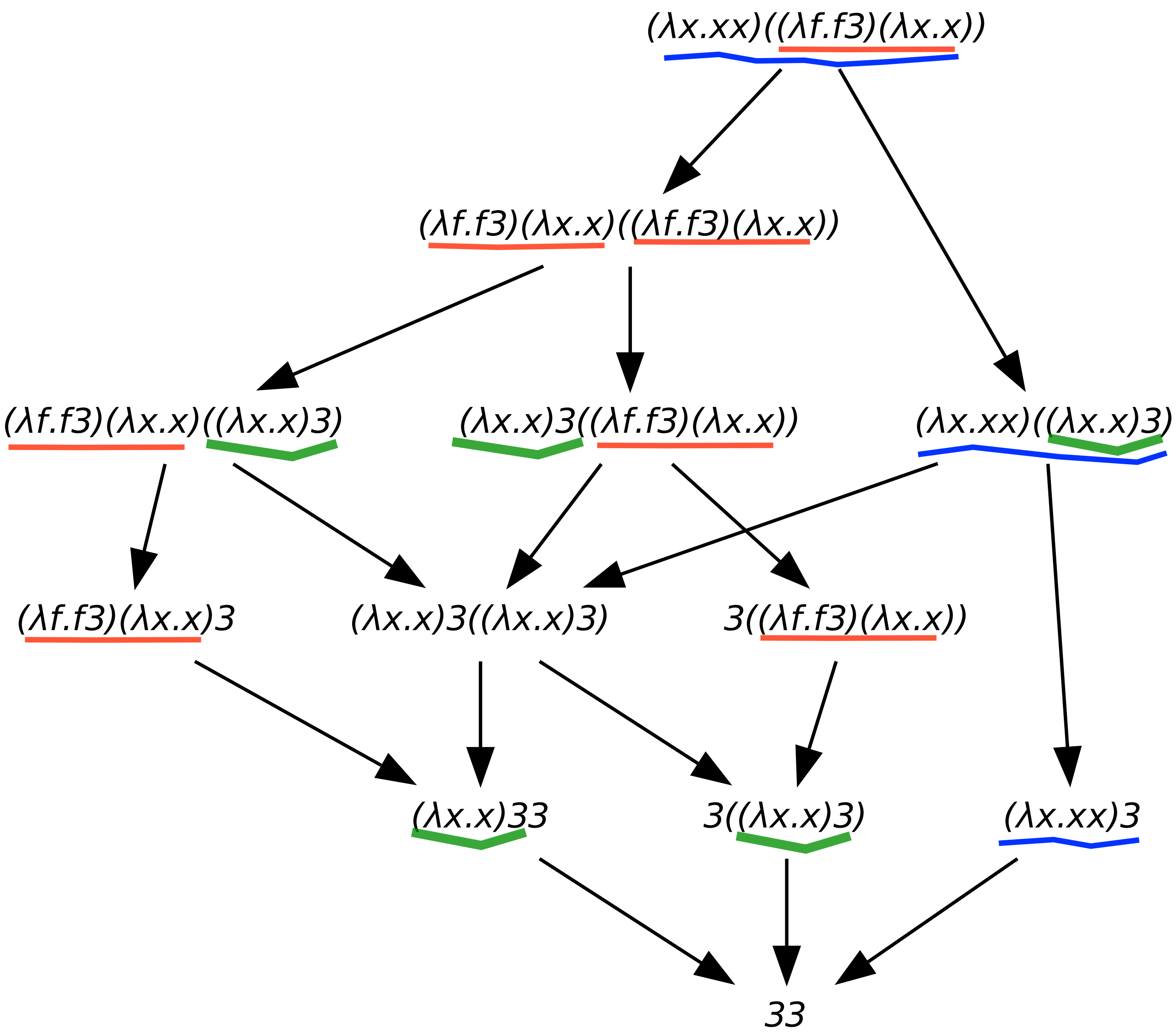
Initial redexes - new redexes



- **Red** and **blue** are initial redexes. **Green** is new.

Redexes and their history (1/3)

- **Notation** `[historical redexes]`

  We write $\langle \rho, R \rangle$ when $\rho : M \xrightarrow{\star} N$ and R is redex in N.

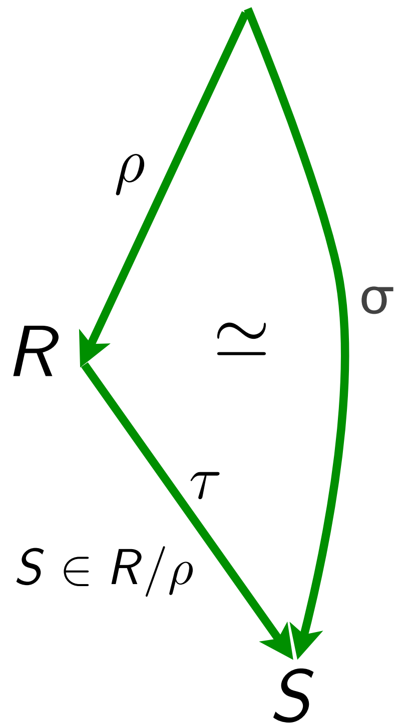
- **Definition** `[copies of redexes]`

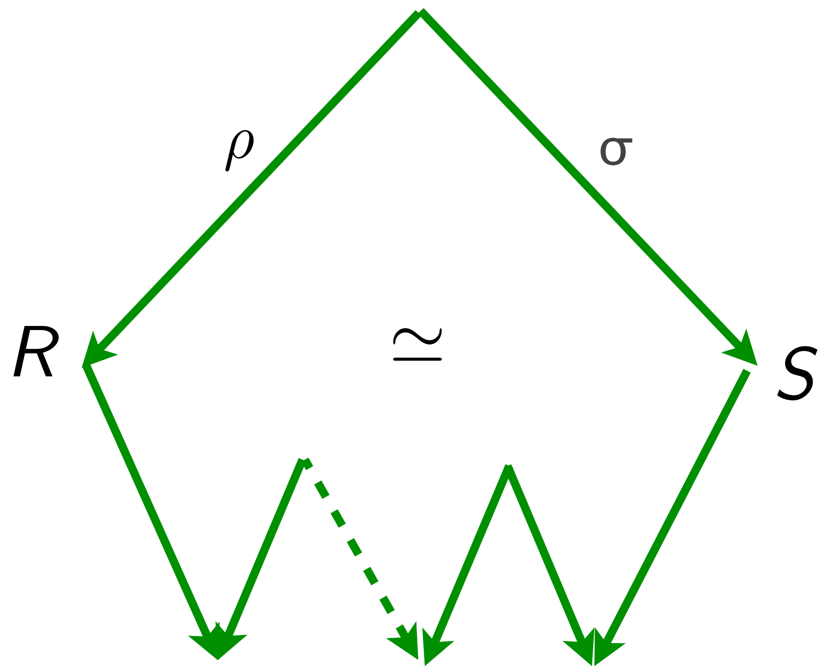
  $\langle \rho, R \rangle \leq \langle \sigma, S \rangle$ when $\rho \sqsubseteq \sigma$ and $S \in R/(\sigma/\rho)$

- **Definition** `[redex families]`

  $\langle \rho, R \rangle \sim \langle \sigma, S \rangle$ stands for the symmetric and transitive closure of the copy relation.

Redexes and their history (2/3)

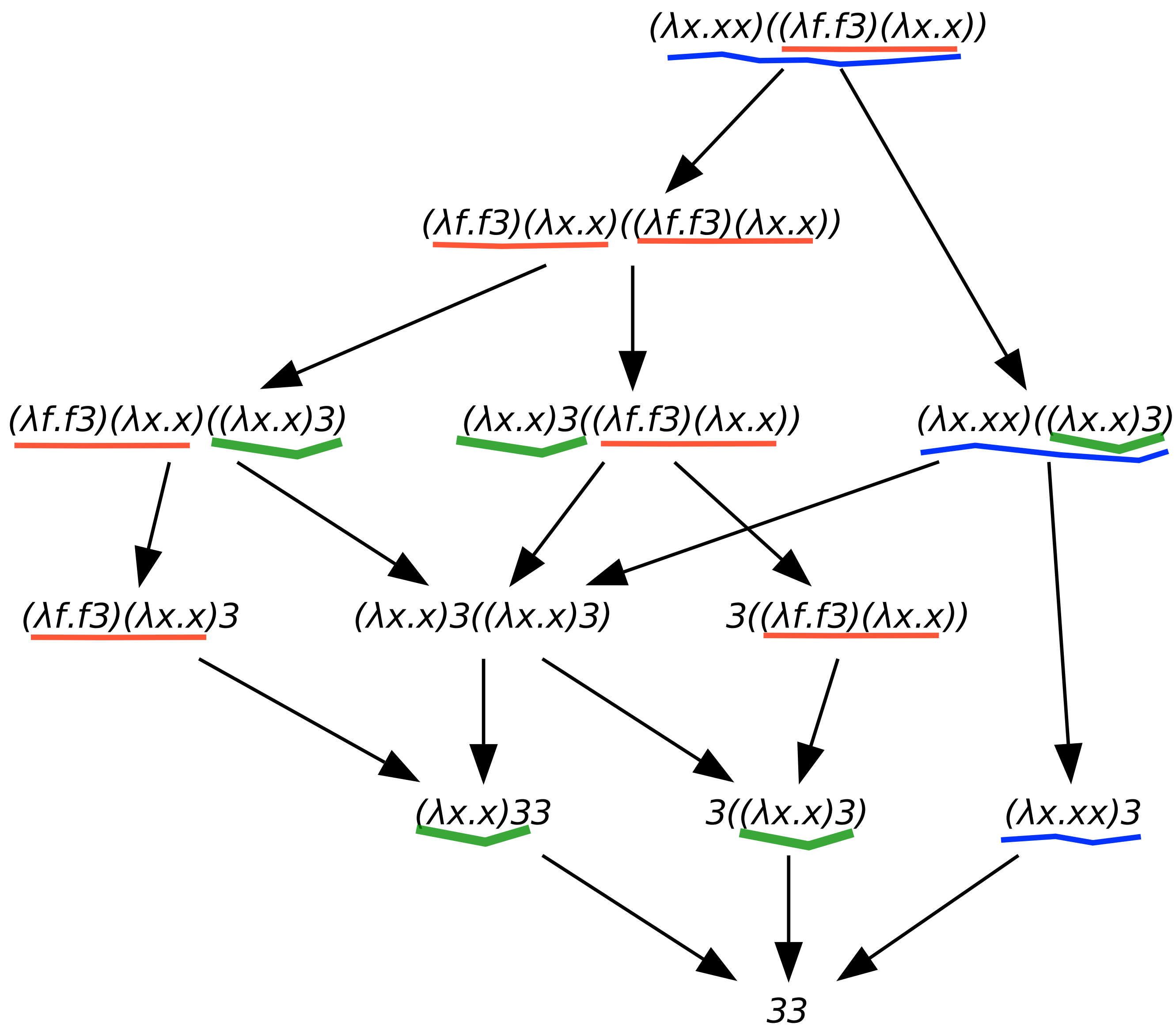




$\langle \rho, R \rangle \leq \langle \sigma, S \rangle$

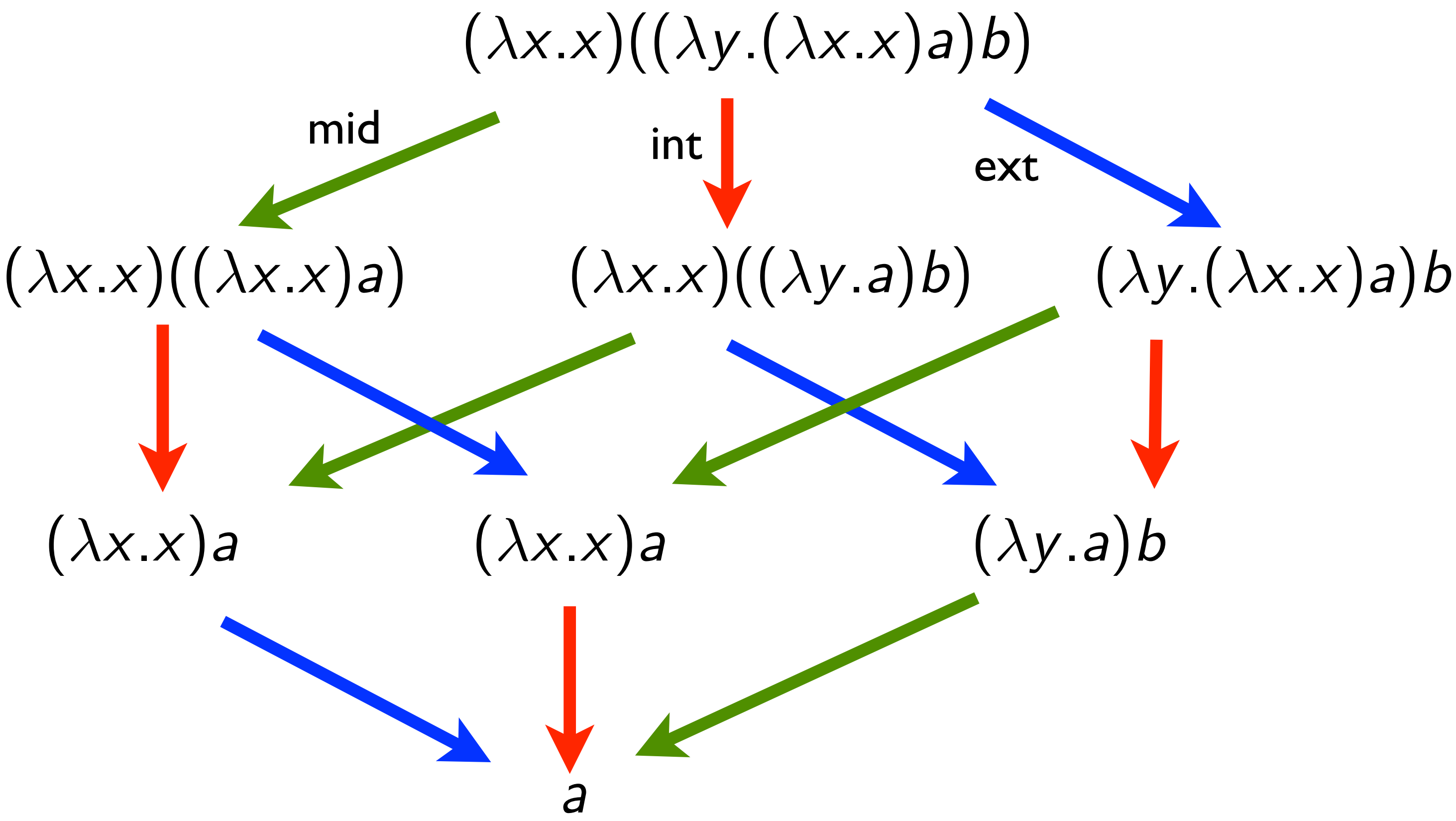
$\langle \rho, R \rangle \sim \langle \sigma, S \rangle$

Redex families (1/3)



- 3 redex families: **red**, **blue**, **green**.

Redex families (2/3)



- 3 redex families: **red**, **blue**, **green**.

Redexes families (3/3)

- **Proposition**

  (a) $T \in R/\rho,\ T \in S/\rho$ implies $R = S$

  (b) $\rho \simeq \sigma$ implies $R/\rho = R/\sigma$

  (c) $\langle \rho, R \rangle \leq \langle \tau, T \rangle,\ \langle \sigma, S \rangle \leq \langle \tau, T \rangle$ implies $\langle \rho, R \rangle \leq \langle \rho \sqcup \sigma, T' \rangle \leq \langle \tau, T \rangle,$
  $\langle \sigma, S \rangle \leq \langle \rho \sqcup \sigma, T' \rangle \leq \langle \tau, T \rangle$

  (d) $\langle \rho, R \rangle \leq \langle \tau, T \rangle,\ \langle \sigma, S \rangle \leq \langle \tau, T \rangle$ does not implies $\langle \tau_0, T_0 \rangle \leq \langle \rho, R \rangle,$
  $\langle \tau_0, T_0 \rangle \leq \langle \sigma, S \rangle$ for some $\langle \tau_0, T_0 \rangle$

  (e) $\langle \rho, R \rangle \sim \langle \sigma, S \rangle$ does not implies $\langle \tau_0, T_0 \rangle \leq \langle \rho, R \rangle,\ \langle \tau_0, T_0 \rangle \leq \langle \sigma, S \rangle$ for some $\langle \tau_0, T_0 \rangle$

  (f) $\langle \rho, R \rangle \sim \langle \sigma, S \rangle$ does not implies $\langle \rho, R \rangle \leq \langle \tau_0, T_0 \rangle,\ \langle \sigma, S \rangle \leq \langle \tau_0, T_0 \rangle$ for some $\langle \tau_0, T_0 \rangle$

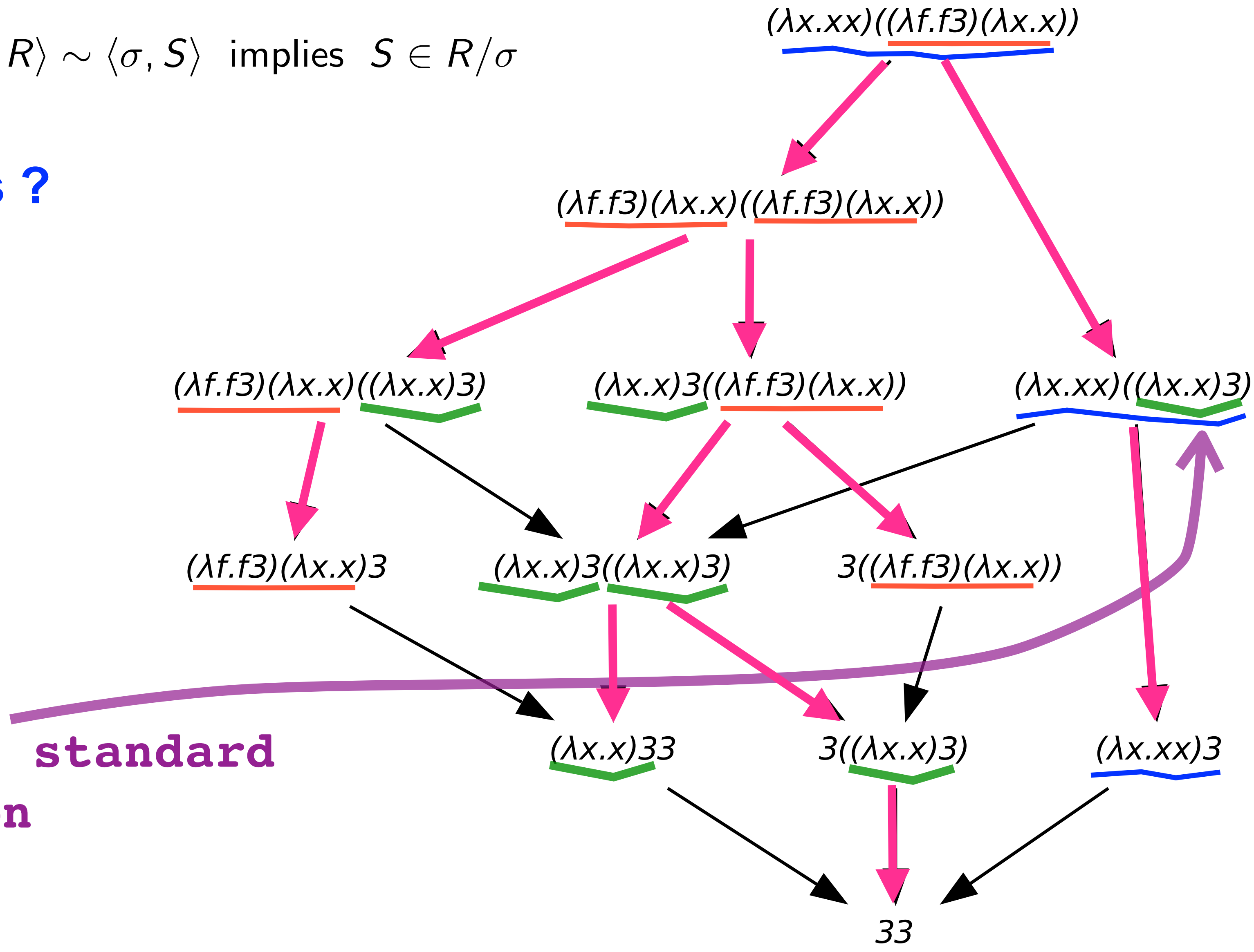
- **Question** Is there a canonical redex in each family ?

Canonical representatives (1/4)

- **Proposition** `[initial redexes]`

$$\langle 0, R\rangle \sim \langle \sigma, S\rangle \text{ implies } S \in R/\sigma$$

- **Others ?**



shortest standard reduction

Canonical representatives (2/4)

- **Definition** `[extraction of canonical redex]`

Let $M = (\lambda x.P)Q\, M_1 M_2 \cdots M_n$ and $\langle \rho_{\mathrm{st}}, R\rangle$ be historical redex from M and H is head redex in M.

$$\mathrm{extract}(H; \rho_{\mathrm{st}}, R) = H; \mathrm{extract}(\rho_{\mathrm{st}}, R)$$

Finite developments

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Parallel steps revisited (1/3)

- parallel steps were defined with inside-out strategy
  `[a la Martin-Löf]`

- can we take any order as reduction strategy ?

- **Definition** A **reduction relative** to a set $\mathcal{F}$ of redexes in M is any reduction contracting only residuals of $\mathcal{F}$.
  A **development** of $\mathcal{F}$ is any maximal relative reduction of $\mathcal{F}$.

Parallel steps revisited (2/3)

- **Theorem** `[Finite Developments, Curry, 50]`

  Let $\mathcal{F}$ be set of redexes in M.

  (1) there are no infinite relative reductions of $\mathcal{F}$,
  (2) they all finish on same term N
  (3) Let R be redex in M. Residuals of R by all finite developments of $\mathcal{F}$ are the same.

- Similar to parallel moves lemma, but we considered particular inside-out reduction strategy.

Parallel steps revisited (3/3)

- **Notation'** `[parallel reduction steps]`
  Let $\mathcal{F}$ be set of redexes in M. We write $M \xrightarrow{\mathcal{F}} N$ if a development of $\mathcal{F}$ connects M to N.

- This notation is consistent with previous results

- Corollaries of FD thm are also parallel moves + cube lemmas

Finite and infinite reductions (1/3)

- **Definition** A **reduction relative** to a set $\mathcal{F}$ of redex families is any reduction contracting redexes in families of $\mathcal{F}$.

  A **development** of $\mathcal{F}$ is any maximal relative reduction.

- **Theorem** `[Finite Developments+, 76]`
  Let $\mathcal{F}$ be a finite set of redex families.

  (1) there are no infinite reductions relative to $\mathcal{F}$,
  (2) they all finish on same term N
  (3) All developments are equivalent by permutations.

Finite and infinite reductions (2/3)

- **Corollary** An **infinite reduction** contracts an **infinite set of redex families.**

- **Corollary** The first-order typed λ-calculus strongly terminates.

Proof In first-order typed λ-calculus:

(1) residuals $R' = (\lambda x.M')N'$ of $R = (\lambda x.M)N$ keep the same type of the function part

(2) new redexes have lower type of their function part

Finite and infinite reductions (3/3)

Proof (cont'd) Created redexes have lower type

$$\underbrace{(\lambda x.\cdots xN\cdots)}_{\sigma\to\tau}\underbrace{(\lambda y.M)}_{\sigma} \to \cdots\underline{(\lambda y.M)N'}\cdots$$

creates

$$\underbrace{(\lambda x.\underbrace{\lambda y.M}_{\tau})}_{\sigma\to\tau}NP \to (\underbrace{\lambda y.M')P}_{\tau}$$

creates

$$\underbrace{(\lambda x.x)}_{\sigma\to\sigma}\underbrace{(\lambda y.M)N}_{\sigma} \to \underbrace{(\lambda y.M)N}_{\sigma}$$

creates

Inside-out reductions

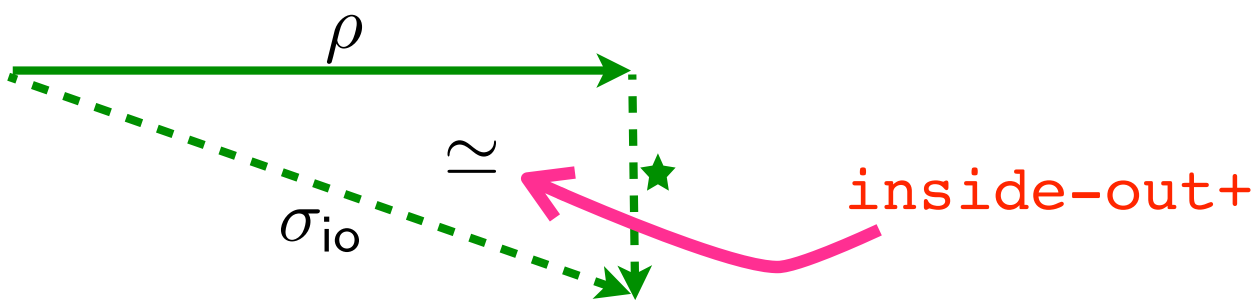
- **Definition:** The following reduction is **inside-out**

$$\rho : M = M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} M_2 \cdots \xrightarrow{R_n} M_n = N$$

iff for all i and j, $i < j$, then R_j is not residual along ρ of some R'_j inside R_i in M_{i-1}.

- **Theorem** [Inside-out completeness, 74]

Let $M \xrightarrow{\star} N$. Then $M \xrightarrow[\mathbf{io}]{\star} P$ and $N \xrightarrow{\star} P$ for some P.



Exercices

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Exercices

- Show