Reductions and Causality (II)

jean-jacques.levy@inria.fr
Tsinghua University,
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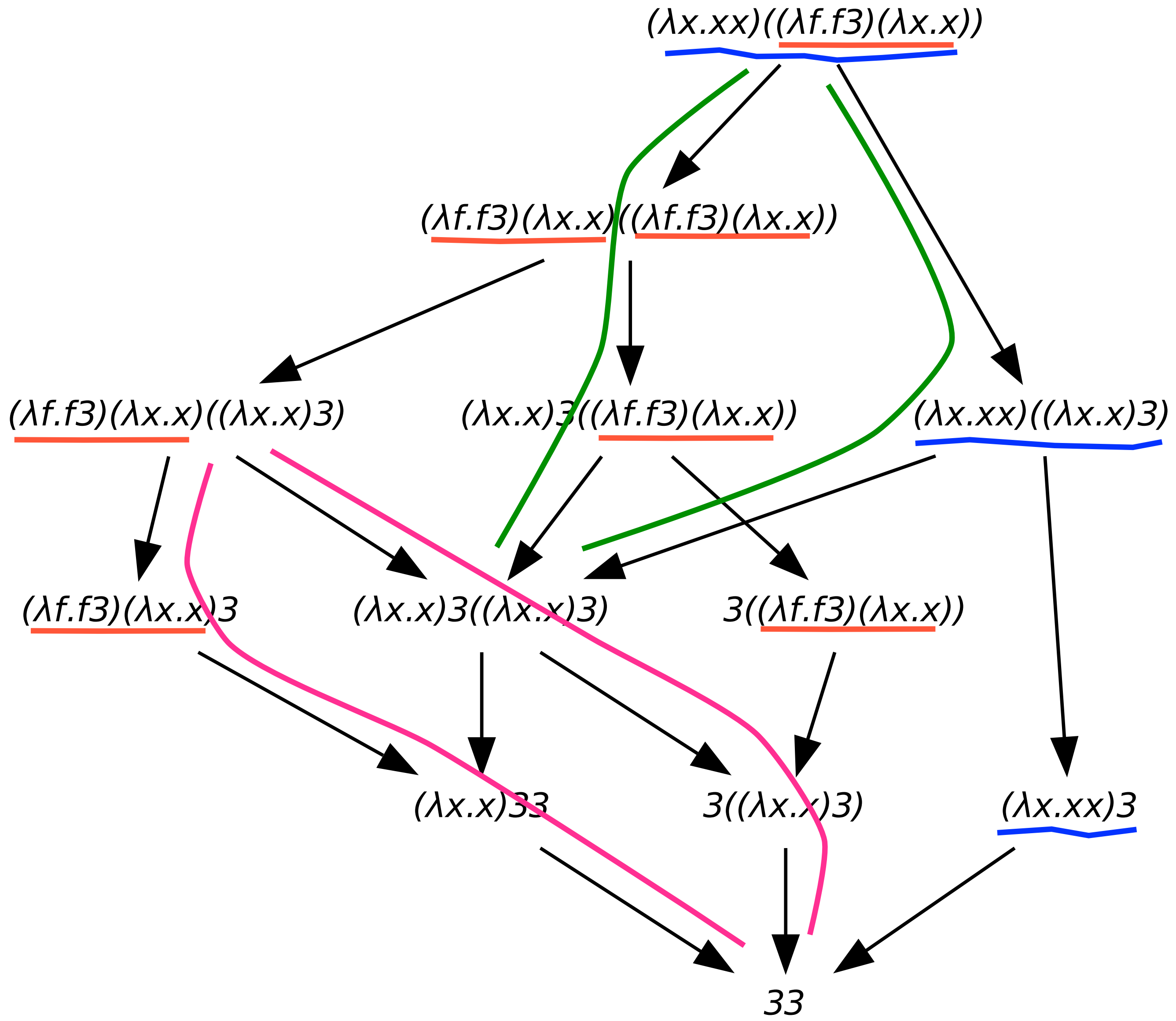
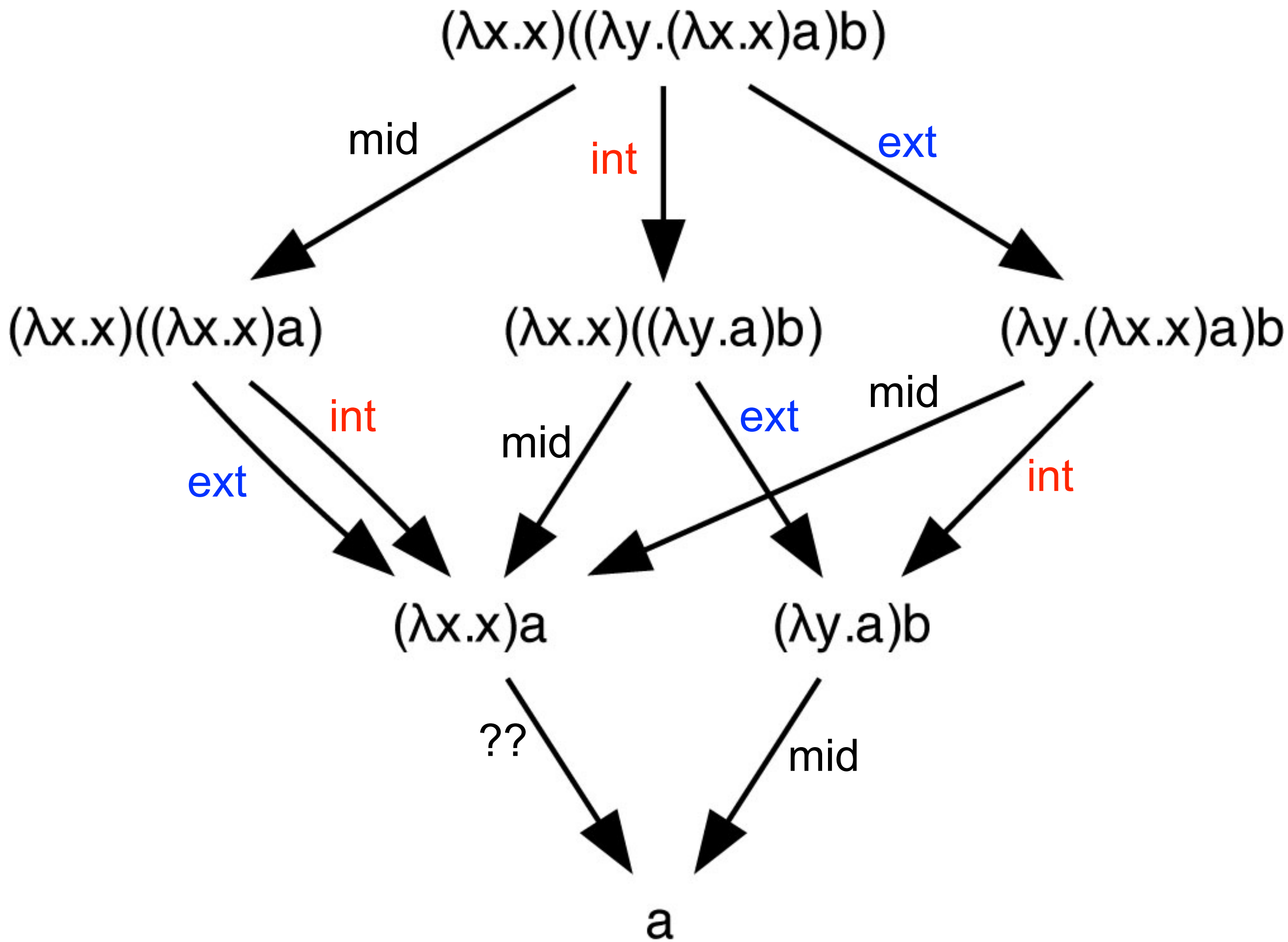
http://pauillac.inria.fr/~levy/courses/tsinghua/reductions

Exercice



(λx.xx)((λf.f3)(λx.x))
(λf.f3)(λx.x)((λf.f3)(λx.x))
(λf.f3)(λx.x)((λx.x)3)
(λx.x)3((λf.f3)(λx.x))
(λx.xx)((λx.x)3)
(λf.f3)(λx.x)3
(λx.x)3((λx.x)3)
3((λf.f3)(λx.x))
(λx.x)33
3((λx.x)3)
(λx.xx)3
33

Exercice



(λx.x)((λy.(λx.x)a)b)
mid
int
ext
(λx.x)((λx.x)a)
(λx.x)((λy.a)b)
(λy.(λx.x)a)b
int
ext
mid
ext
mid
int
(λx.x)a
(λy.a)b
??
mid
a

Plan

- parallel reduction steps
- cube lemma
- residuals of reductions
- equivalence by permutations
- beyond the λ-calculus

Parallel reduction steps

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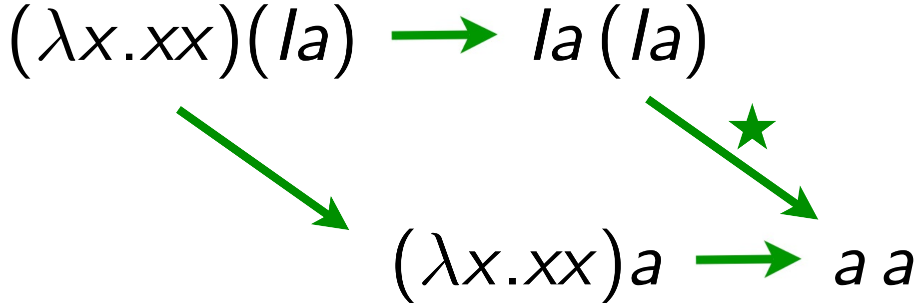
Parallel reductions (1/3)

- permutation of reductions has to cope with copies of redexes



- in fact, a parallel reduction $Ia(Ia) \not\longrightarrow aa$

- in λ-calculus, need to define parallel reductions for nested sets

Parallel reductions (2/3)

- the axiomatic way (à la Martin-Löf)

[Var Axiom] $x \twoheadrightarrow\!\!\!\!\!/\!\!/\;\; x$ [Const Axiom] $c \twoheadrightarrow\!\!\!\!\!/\!\!/\;\; c$

$$\text{[App Rule]}\ \frac{M \twoheadrightarrow\!\!\!\!\!/\!\!/\;\; M' \quad N \twoheadrightarrow\!\!\!\!\!/\!\!/\;\; N'}{MN \twoheadrightarrow\!\!\!\!\!/\!\!/\;\; M'N'} \qquad \text{[Abs Rule]}\ \frac{M \twoheadrightarrow\!\!\!\!\!/\!\!/\;\; M'}{\lambda x.M \twoheadrightarrow\!\!\!\!\!/\!\!/\;\; \lambda x.M'}$$

$$\text{[//Beta Rule]}\ \frac{M \twoheadrightarrow\!\!\!\!\!/\!\!/\;\; M' \quad N \twoheadrightarrow\!\!\!\!\!/\!\!/\;\; N'}{(\lambda x.M)N \twoheadrightarrow\!\!\!\!\!/\!\!/\;\; M'\{x := N'\}}$$

- example:

$$(\lambda x.Ix)(Iy) \twoheadrightarrow\!\!\!\!\!/\!\!/\;\; (\lambda x.x)y$$

$$(\lambda x.(\lambda y.yy)x)(Ia) \twoheadrightarrow\!\!\!\!\!/\!\!/\;\; Ia(Ia)$$

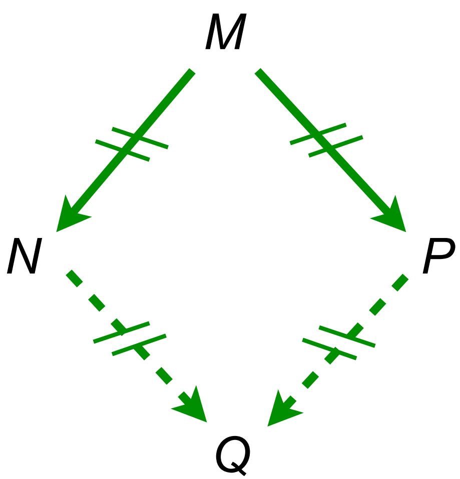
$$(\lambda x.(\lambda y.yy)x)(Ia) \twoheadrightarrow\!\!\!\!\!/\!\!/\;\; (\lambda y.yy)a$$

- it's an *inside-out* parallel reduction

Parallel reductions (3/3)

- **Parallel moves lemma** `[Curry 50]`

If $M \mathrel{\rightarrow\!\!\!\!\!\!\!/\!\!/\,} N$ and $M \mathrel{\rightarrow\!\!\!\!\!\!\!/\!\!/\,} P$, then $N \mathrel{\rightarrow\!\!\!\!\!\!\!/\!\!/\,} Q$ and $P \mathrel{\rightarrow\!\!\!\!\!\!\!/\!\!/\,} Q$ for some Q.



lemma 1-1-1-1
(strong confluency)

- Enough to prove Church Rosser thm since $\rightarrow \subset \mathrel{\rightarrow\!\!\!\!\!\!\!/\!\!/\,} \subset \stackrel{\star}{\rightarrow}$

`[Tait--Martin Löf 60?]`

// Reductions of set of redexes (1/4)

- Goal: parallel reduction of a given set of redexes

$$M, N ::= x \mid \lambda x.M \mid MN \mid (\lambda x.M)^a N$$

$$a, b, c, \ldots ::= \text{redex labels}$$

$$(\lambda x.M)N \longrightarrow M\{x := N\}$$

$$(\lambda x.M)^a N \longrightarrow M\{x := N\}$$

- Substitution as before with add-on:

$$((\lambda y.P)^a Q)\{x := N\} = (\lambda y.P\{x := N\})^a Q\{x := N\}$$

// Reductions of set of redexes (2/4)

- let $\mathcal{F}$ be a set of redex labels in M

$$\text{[Var Axiom]}\ x \xrightarrow{\mathcal{F}} x \qquad\qquad \text{[Const Axiom]}\ c \xrightarrow{\mathcal{F}} c$$

$$\text{[App Rule]}\ \frac{M \xrightarrow{\mathcal{F}} M' \quad N \xrightarrow{\mathcal{F}} N'}{MN \xrightarrow{\mathcal{F}} M'N'} \qquad\qquad \text{[Abs Rule]}\ \frac{M \xrightarrow{\mathcal{F}} M'}{\lambda x.M \xrightarrow{\mathcal{F}} \lambda x.M'}$$

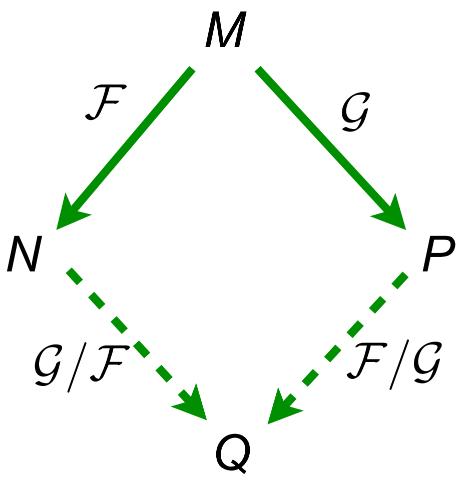
$$\text{[//Beta Rule]}\ \frac{M \xrightarrow{\mathcal{F}} M' \quad N \xrightarrow{\mathcal{F}} N' \quad a \in \mathcal{F}}{(\lambda x.M)^a N \xrightarrow{\mathcal{F}} M'\{x := N'\}} \qquad \text{[Redex']}\ \frac{M \xrightarrow{\mathcal{F}} M' \quad N \xrightarrow{\mathcal{F}} N' \quad a \notin \mathcal{F}}{(\lambda x.M)^a N \xrightarrow{\mathcal{F}} (\lambda x.M')^a N'}$$

- let $\mathcal{F}$, $\mathcal{G}$ be set of redexes in M and let $M \xrightarrow{\mathcal{F}} N$, then the set $\mathcal{G}/\mathcal{F}$ of residuals of $\mathcal{G}$ by $\mathcal{F}$ is the set of $\mathcal{G}$ redexes in N.

// Reductions of set of redexes (3/4)

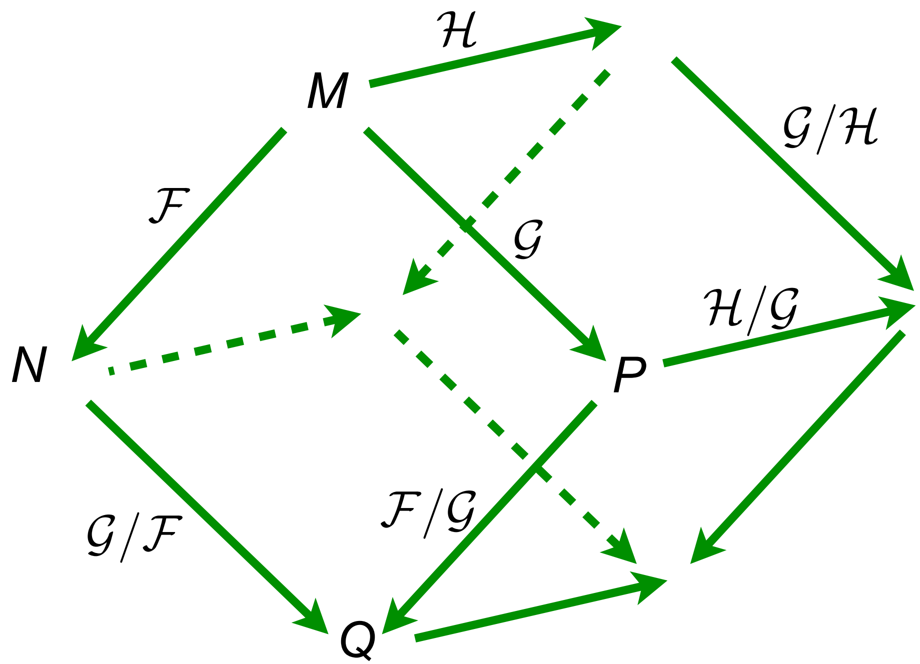
- **Parallel moves lemma+** [Curry 50]

If $M \xrightarrow{\mathcal{F}} N$ and $M \xrightarrow{\mathcal{G}} P$, then $N \xrightarrow{\mathcal{G}/\mathcal{F}} Q$ and $P \xrightarrow{\mathcal{F}/\mathcal{G}} Q$ for some Q.



// Reductions of set of redexes (4/4)

- **Parallel moves lemma++** `[Curry 50] The Cube Lemma`



- Then $(\mathcal{H}/\mathcal{F})/(\mathcal{G}/\mathcal{F}) = (\mathcal{H}/\mathcal{G})/(\mathcal{F}/\mathcal{G})$

Residuals of reductions

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Parallel reductions

- Redex occurences and labels
  - Let $||U|| = M$ where labels in U are erased (forgetful functor)
  - Then $M \xrightarrow{\mathcal{F}} N$ iff $U \xrightarrow{\mathcal{F}} N$ for some labeled U and $M = ||U||$

- Consider reductions where each step is parallel

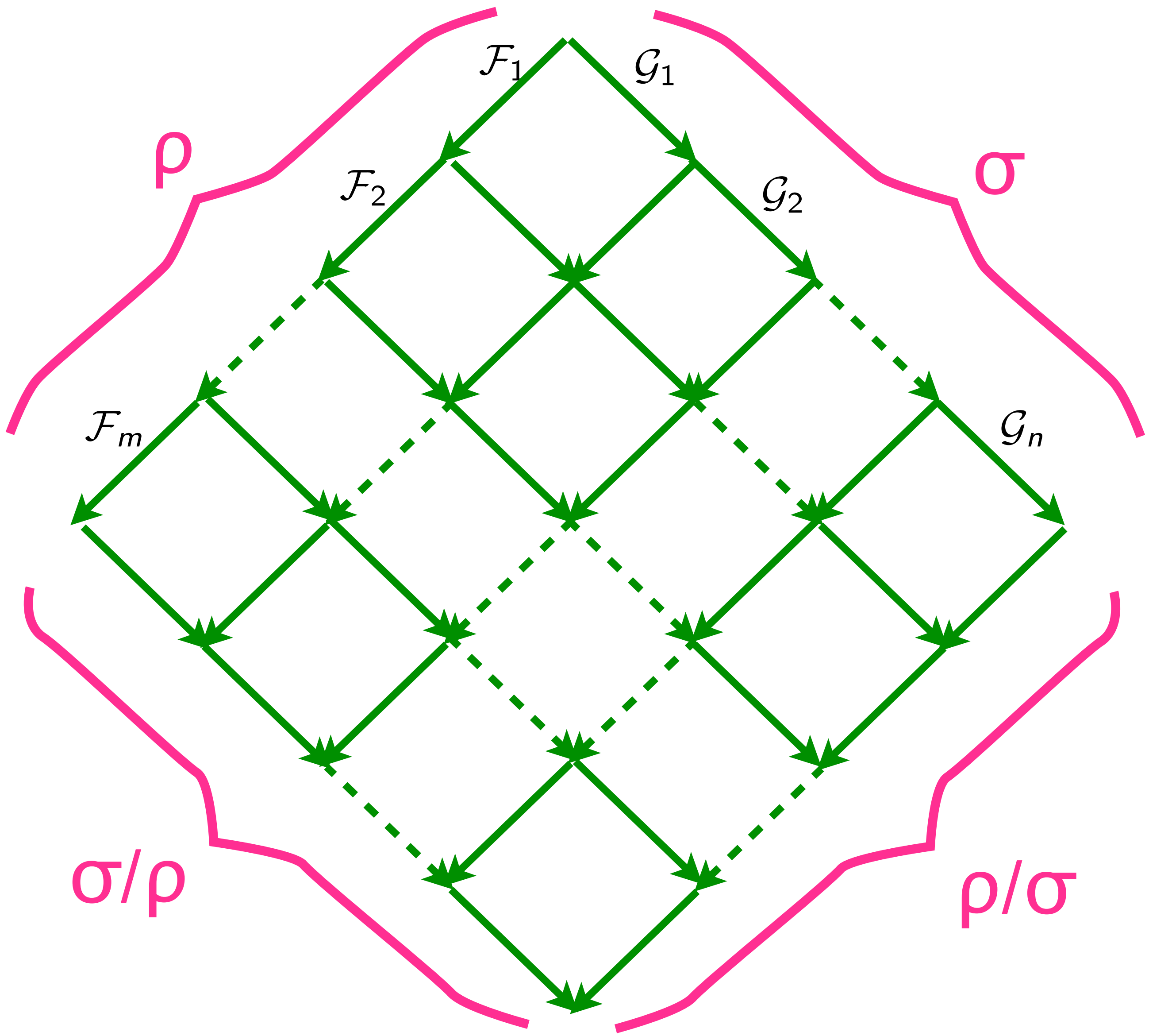
$$\rho : M = M_0 \xrightarrow{\mathcal{F}_1} M_1 \xrightarrow{\mathcal{F}_2} M_2 \cdots \xrightarrow{\mathcal{F}_n} M_n = N$$

- We also write

$$\rho = 0 \text{ when } n = 0$$

$$\rho = \mathcal{F}_1; \mathcal{F}_2; \cdots \mathcal{F}_n \text{ when } M \text{ clear from context}$$

Residual of reductions (1/4)



Residual of reductions (2/4)

- **Definition** [JJ 76]

  $$\rho/0 = \rho$$

  $$\rho/(\sigma;\tau) = (\rho/\sigma)/\tau$$

  $$(\rho;\sigma)/\tau = (\rho/\tau);(\sigma/(\tau/\rho))$$

  $\mathcal{F}/\mathcal{G}$ already defined

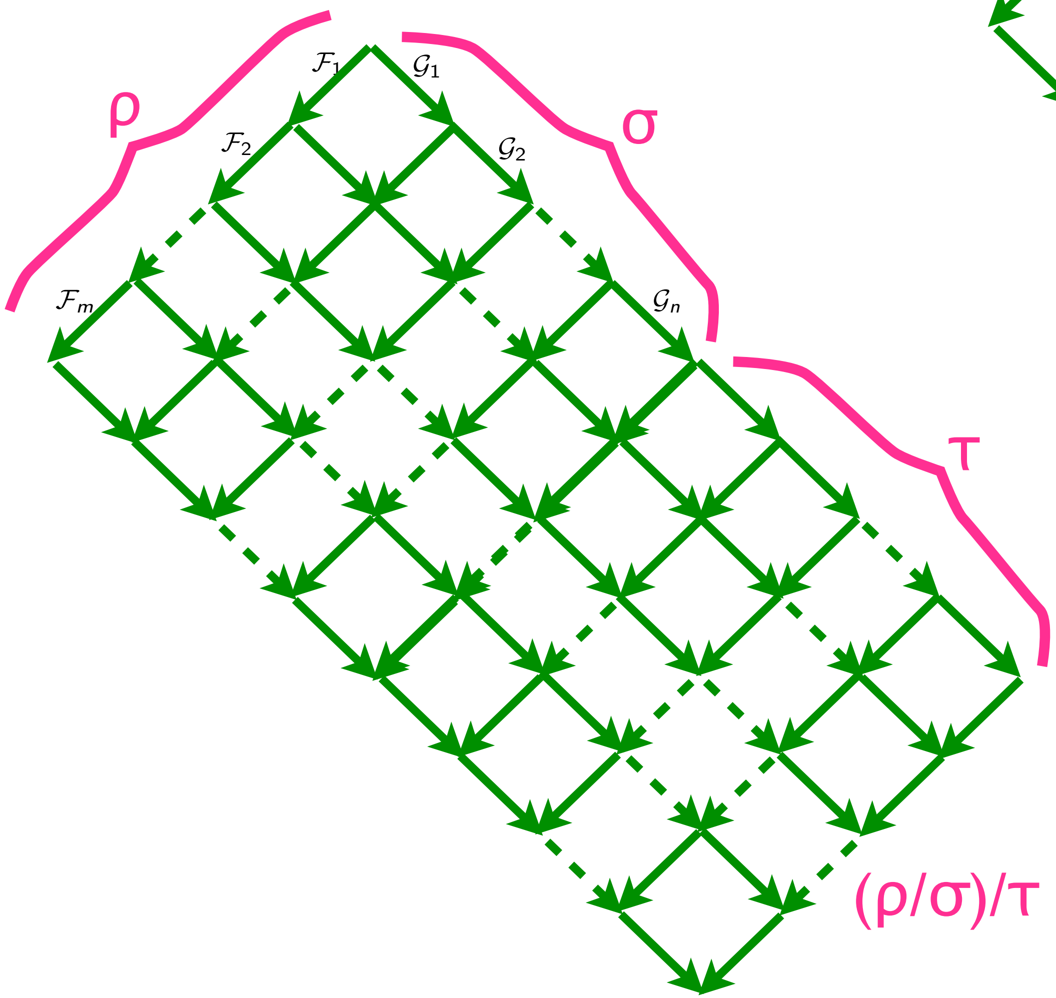
- **Notation**

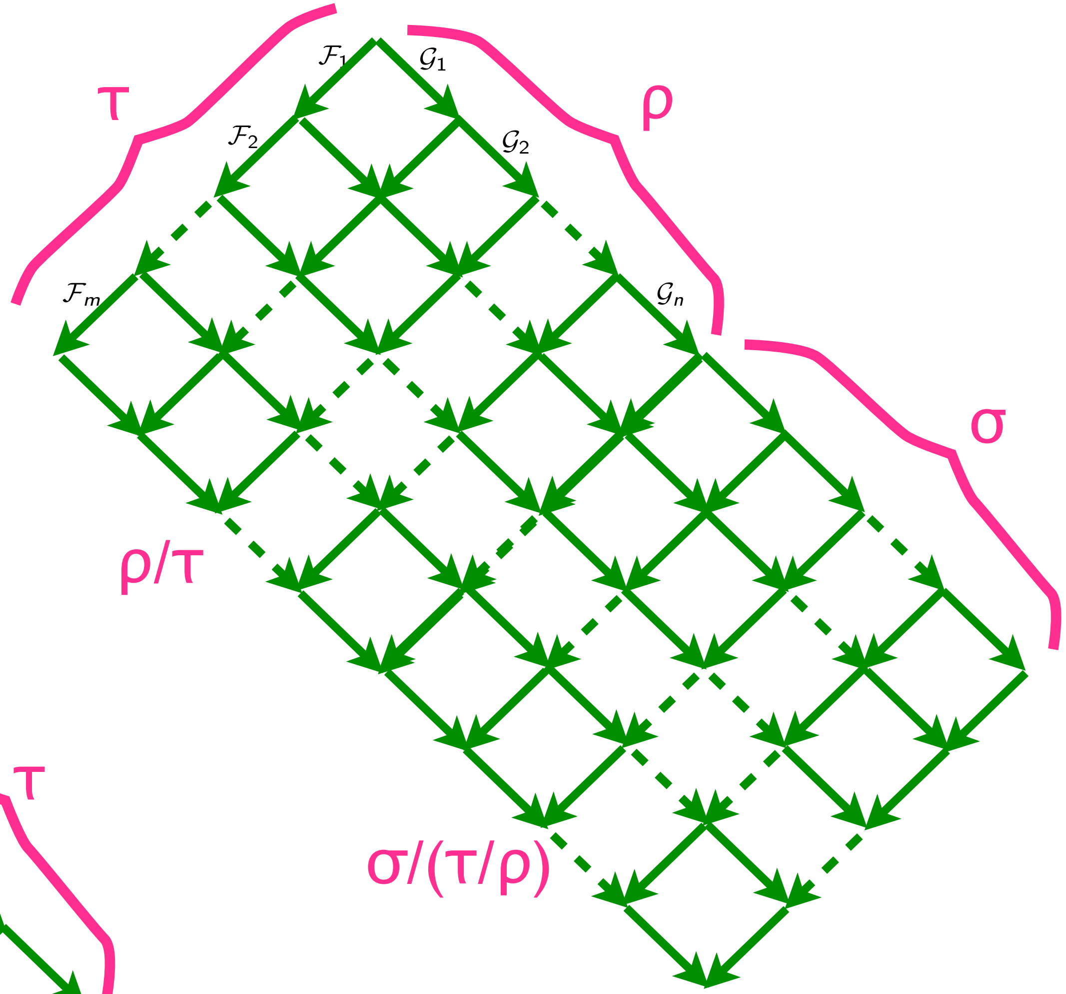
  $$\rho \sqcup \sigma = \rho;(\sigma/\rho)$$

- **Proposition** [Parallel Moves +]:

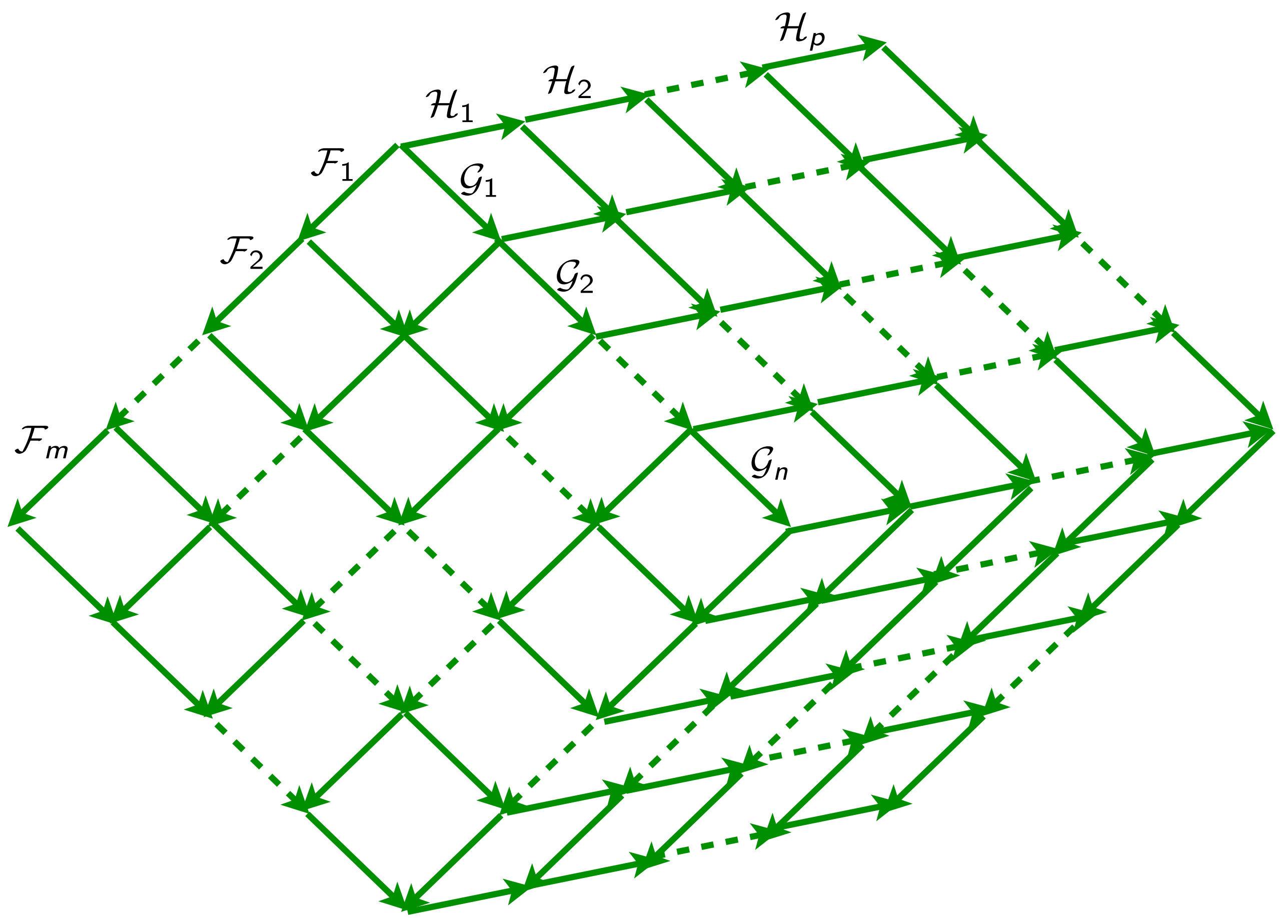
  $\rho \sqcup \sigma$ and $\sigma \sqcup \rho$ are cofinal

Residual of reductions (3/4)





Residual of reductions (4/4)



- **Proposition** `[Cube Lemma ++]`:

$$\tau/(\rho \sqcup \sigma) = \tau/(\sigma \sqcup \rho)$$

Equivalence by permutations

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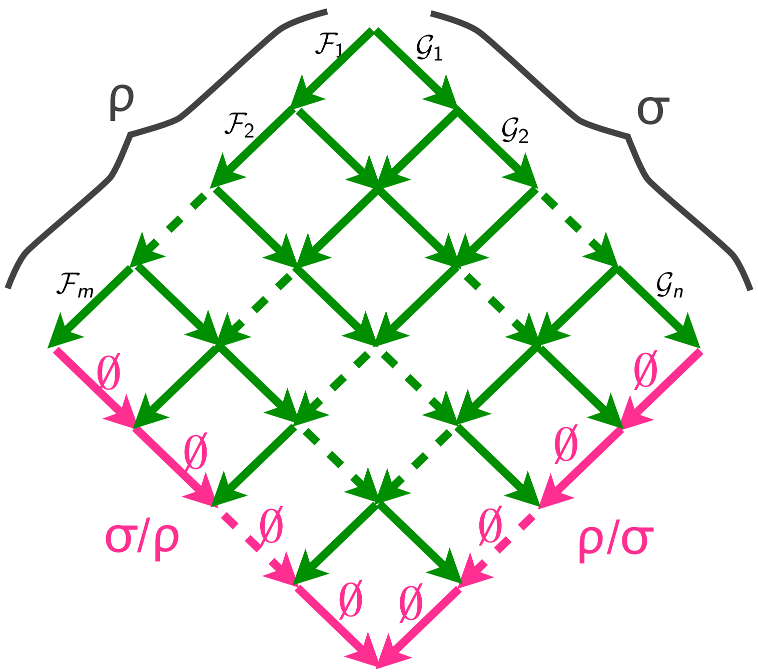
Equivalence by permutations (1/4)

- **Definition:**

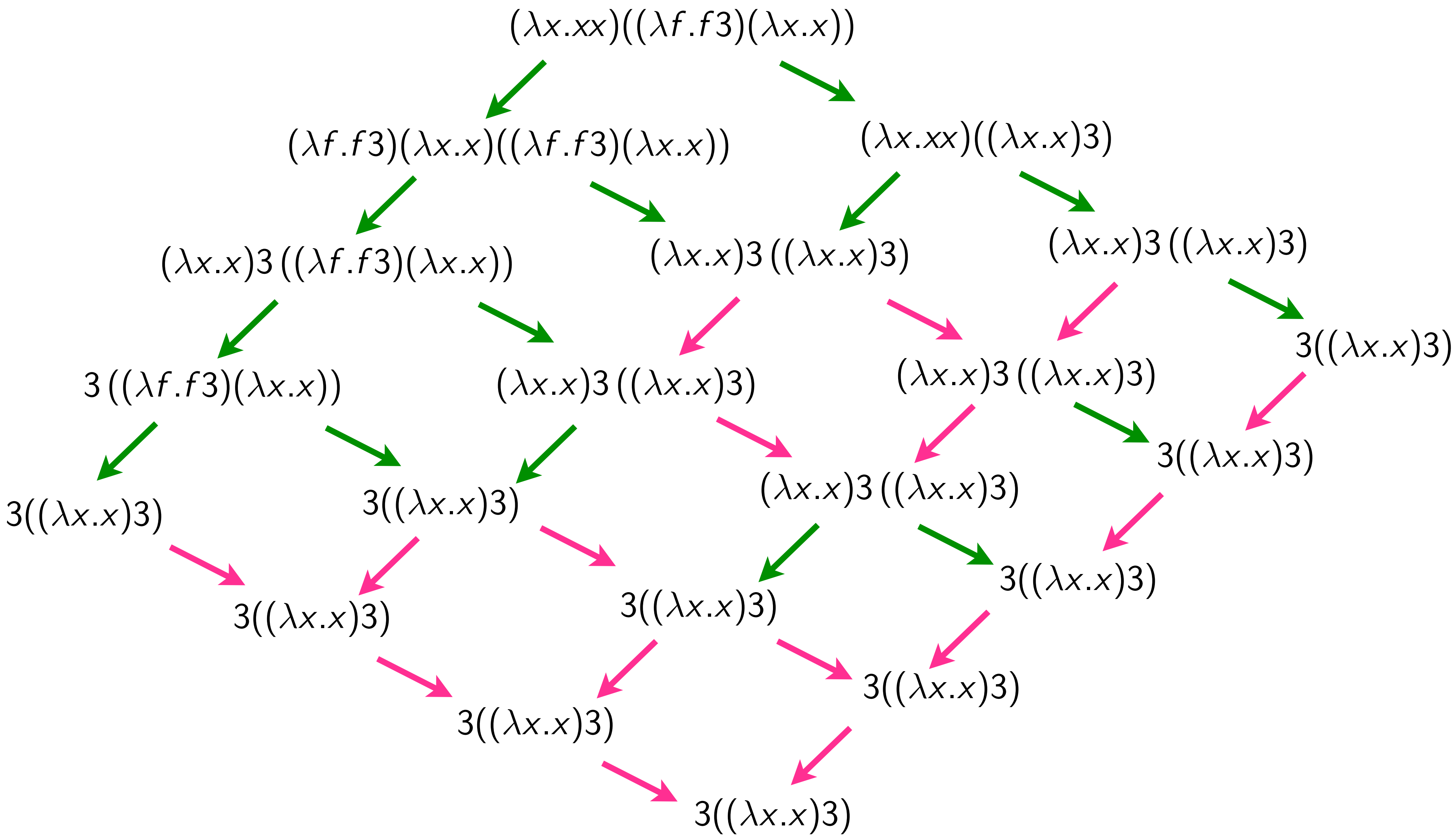
  Let ρ and σ be 2 coinitial reductions. Then ρ is equivalent to σ by permutations, $\rho \simeq \sigma$, iff:

$$\rho/\sigma = \emptyset^m \quad \text{and} \quad \sigma/\rho = \emptyset^n$$

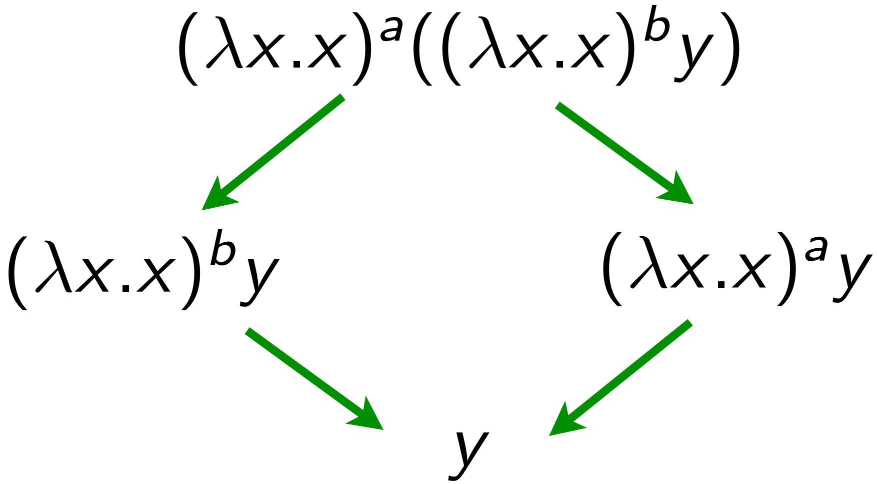


- Notice that $\rho \simeq \sigma$ means that ρ and σ are cofinal

Equivalence by permutations (2/4)



Equivalence by permutations (3/4)



$$\rho : M = I(Iy) \xrightarrow{R_a} Iy = N$$

$$\sigma : M = I(Iy) \xrightarrow{R_b} Iy = N$$

$$\rho \not\simeq \sigma$$

- Notice that $\rho \not\simeq \sigma$ while ρ and σ are coinitial and cofinal

Equivalence by permutations (4/4)

- Same with $0 \not\simeq \rho$ when $\rho : \Delta\Delta \longrightarrow \Delta\Delta$

  $\Delta = \lambda x.xx$

- **Exercice 1**: Give other examples of non-equivalent reductions between same terms

- **Exercice 2**: Show following equalities

$$\rho/0 = \rho \qquad \emptyset^n/\rho = \emptyset^n$$
$$0/\rho = 0 \qquad 0 \simeq \emptyset^n$$
$$\rho/\emptyset^n = \rho \qquad \rho/\rho = \emptyset^n$$

- **Exercice 3**: Show that $\simeq$ is an equivalence relation.

Properties of equivalent reductions

- **Proposition**

$$\rho \simeq \sigma \quad \text{iff} \quad \forall \tau,\ \tau/\rho = \tau/\sigma$$

$$\rho \simeq \sigma \text{ implies } \rho/\tau \simeq \sigma/\tau$$

$$\rho \simeq \sigma \quad \text{iff} \quad \tau;\rho \simeq \tau;\sigma$$

$$\rho \simeq \sigma \quad \text{implies} \quad \rho;\tau \simeq \sigma;\tau$$

$$\rho \sqcup \sigma \simeq \sigma \sqcup \rho$$

- **Proof**

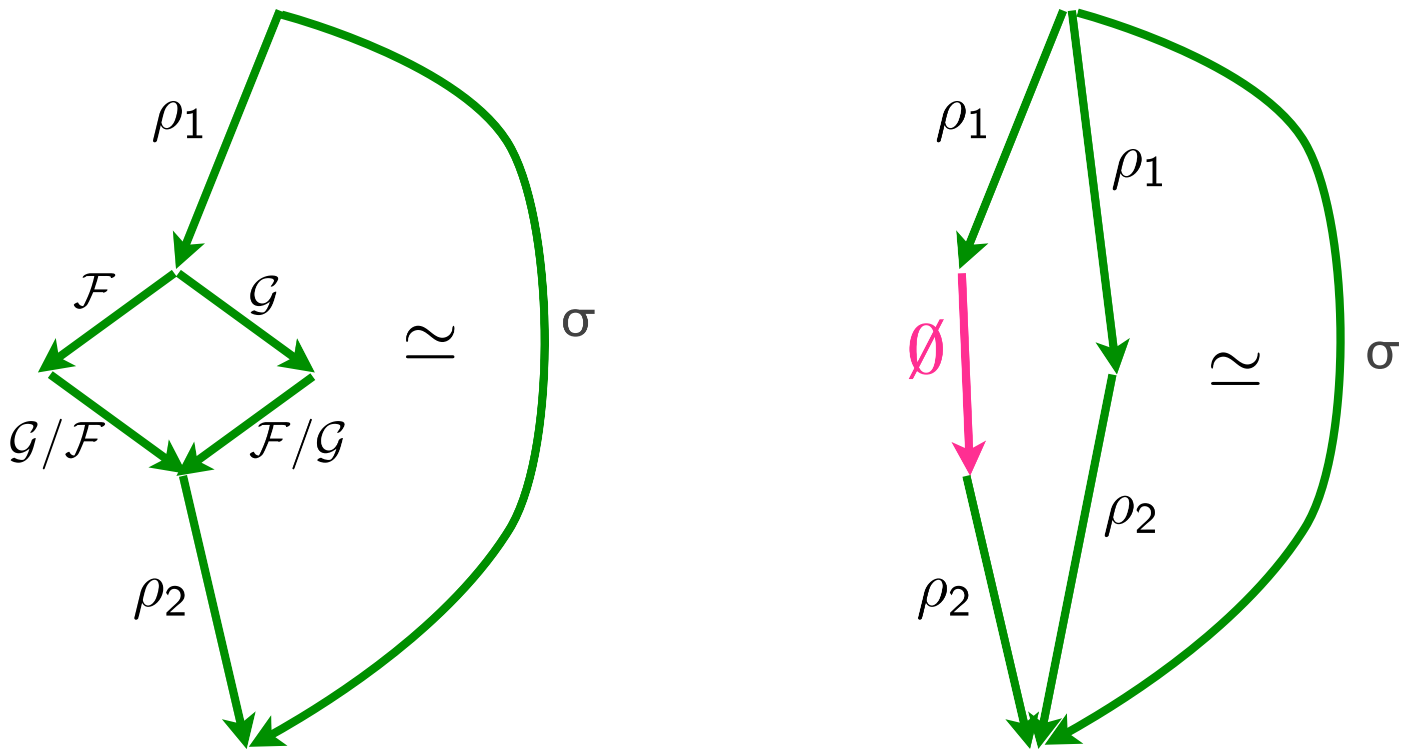
As $\rho \simeq \sigma$, one has $\sigma/\rho = \emptyset^n$. Therefore $\tau/\rho = (\tau/\rho)/(\sigma/\rho)$. That is $\tau/\rho = \tau/(\rho \sqcup \sigma)$. Similarly as $\sigma \simeq \rho$, one gets $\tau/\sigma = \tau/(\sigma \sqcup \rho)$. But cube lemma says $\tau/(\rho \sqcup \sigma) = \tau/(\sigma \sqcup \rho)$. Therefore $\tau/\rho = \tau/\sigma$.

Properties of equivalent reductions

- **Proposition** $\simeq$ is the smallest congruence containing

$$\mathcal{F};(\mathcal{G}/\mathcal{F}) \simeq \mathcal{G};(\mathcal{F}/\mathcal{G})$$

$$0 \simeq \emptyset$$



Beyond λ-calculus

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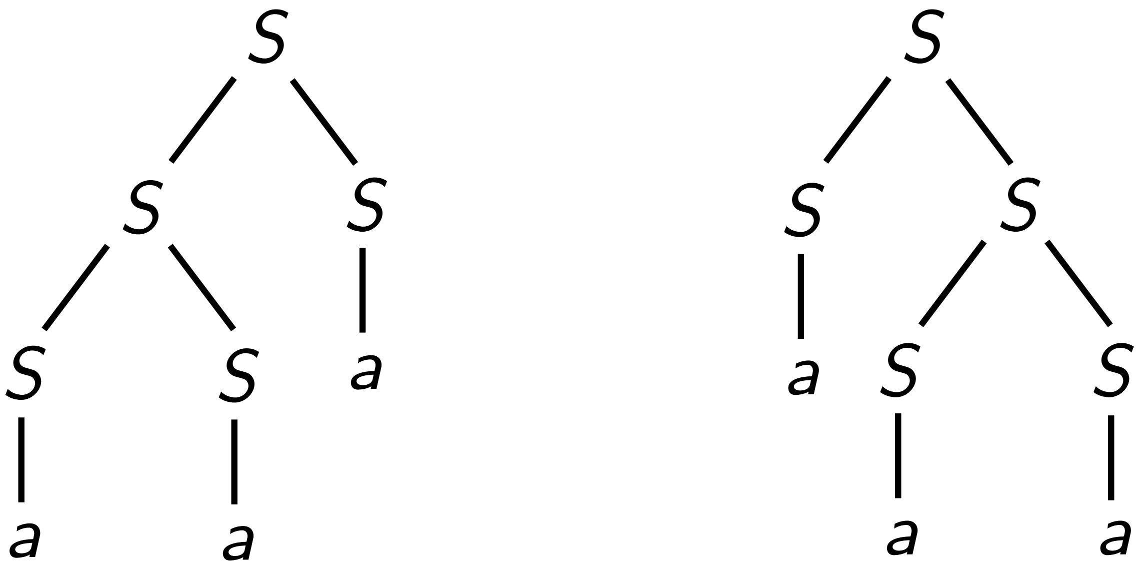
Context-free languages

- permutations of derivations in contex-free languages

$$S \to SS$$

$$S \to a$$



- each parse tree corresponds to an equivalence class

Term rewriting

- permutations of derivations are defined with critical pairs
- critical pairs make conflicts
- only 2nd definition of equivalence works [Boudol, 1982]

Process algebras

- similar to TRS [Boudol-Castellani, 1982]

Exercices

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Exercices

- **Exercice 4**: Complete all proofs of propositions
- **Exercice 5**: Show equivalent reductions in