Lambda-Calculus (III-4)

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Consistent theories

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Plan

- Consistent lambda theories
- Extensional equivalences
- Congruences and semantics
- Bohm trees

Consistency

- A **lambda-theory** is any congruence containing β-equality (interconvertiblity)
- More precisely, a lambda-theory satisfies the following axioms and rules:

$$x \equiv x \qquad c \equiv c \qquad (\lambda x.M)N \equiv M\{x := N\}$$

$$\frac{M \equiv M'}{MN \equiv M'N} \qquad \frac{N \equiv N'}{MN \equiv MN'} \qquad \frac{M \equiv M'}{\lambda x.M \equiv \lambda x.M'}$$

- A lambda-theory is **consistent** iff $M \not\equiv N$ for some M, N.

Exercice 1

1- Give examples of consistent theories.

2- Show that any lambda-theory containing $x \equiv y$ is inconsistent when $x \neq y$.

3- Same with $I \equiv K$.

Extensional theories

- An extensional lambda-theory satisfies the η-rule.

$$x \equiv x \qquad c \equiv c \qquad (\lambda x.M)N \equiv M\{x := N\}$$

$$\frac{M \equiv M'}{MN \equiv M'N} \qquad \frac{N \equiv N'}{MN \equiv MN'} \qquad \frac{M \equiv M'}{\lambda x.M \equiv \lambda x.M'}$$

$$\lambda x.Mx \equiv M \quad (x \notin \mathrm{var}(M))$$

Exercice 2

- Show previous definition is equivalent to following:

$$x \equiv x \qquad c \equiv c \qquad (\lambda x.M)N \equiv M\{x := N\}$$

$$\frac{M \equiv M'}{MN \equiv M'N} \qquad \frac{N \equiv N'}{MN \equiv MN'} \qquad \frac{M \equiv M'}{\lambda x.M \equiv \lambda x.M'}$$

$$\frac{\forall P.\ MP \equiv NP}{M \equiv N}$$

Contexts

- A context C[ ] is a λ-term with a hole. More precisely:

$$C[\,] \quad ::= \quad [\,] \mid C[\,]N \mid MC[\,] \mid \lambda x.C[\,]$$

- By *C[M]*, we mean the λ-term obtained by putting M in the hole.

- A λ-theory is any equivalence relation $\equiv$ satisfying:

$$M \longrightarrow N \Rightarrow M \equiv N$$

$$M \equiv N \Rightarrow C[M] \equiv C[N]$$

What are consistent λ-theories ?

- Can we equate 2 different normal forms ?

- No by Bohm theorem!

- **Theorem (Böhm)[1968]** Let M and N be two normals forms such that $M \not\equiv_\eta N$. Let x and y be two variables. There exists a context $C[\,]$ such that:

$$C[M] \xrightarrow{*} x$$

$$C[N] \xrightarrow{*} y$$

Proof: not easy !!

- **Corollary:** any λ-theory equating two different normal forms is inconsistent.

Proof: easy ! Do it as exercice.

What are consistent λ-theories ?

- Can we equate all terms without normal forms ?

- No by a similar argument !

- **Fact:**

Take $M = x(\Delta\Delta)I$ and $N = x(\Delta\Delta)K$.

Then M and N have no normal forms. Thus $M \equiv N$ and $C[M] \equiv C[N]$ in any context $C[\,]$.

Take $C[\,] = (\lambda x.[\,](KI))$. Then $C[M] \xrightarrow{*} KI(\Delta\Delta)I \xrightarrow{*} I$. And $C[N] \xrightarrow{*} KI(\Delta\Delta)K \xrightarrow{*} K$.

Therefore $I \equiv C[M] \equiv C[N] \equiv K$. Which is not consistent.

- **Exercice** Do similar argument with $xI(\Delta\Delta) \equiv x(\Delta\Delta)I$

Head normal forms

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Total undefinedness

- A term *M* is **totally undefined** iff for all context *C[ ]* whenever there exists *N* such that *C[N]* has no normal form, then *C[M]* has no normal form.

- Thus *M* is totally undefined iff for all context *C[ ]* when *C[M]* has a normal form, then *C[N]* has also a normal form for every *N*.

- Examples:

  1- $x(\Delta\Delta)I$ is not totally undefined, since $(\lambda x.x(\Delta\Delta)I)(KI)$ has a normal form, but not $(\lambda x.\Delta\Delta)(KI)$.

  2- $xI(\Delta\Delta)$ is not totally undefined, by similar argument.

  3- $\Delta\Delta$ is totally undefined. Proof is a bit complex. Intuitively, if $C[\Delta\Delta]$ has a normal form, one can reach it by the leftmost-outermost reduction. Never a residual of $\Delta\Delta$ is contracted in this reduction, since it would have been an endless leftmost-outermost redex and this normal reduction would not get the normal form. Then by plugging any N in place of $\Delta\Delta$ in initial term, one get the same reduction and ends with same normal form.

Head normal forms

- Fortunately, there is another (intensional) characterization of totally undefined terms .

- A term is in **head normal form** (hnf) iff it has the following form:

  $\lambda x_1 x_2 \cdots x_m . x M_1 M_2 \cdots M_n$ with $m \geq 0$ and $n \geq 0$

  head variable

  (x may be free or bound by one of the x_i)

- A term not in **head normal form** is of following form:

  $\lambda x_1 x_2 \cdots x_m . (\lambda x.M) N N_1 N_2 \cdots N_n$

  head redex

- Head normal forms appeared in **Wadsworth's** phD [1973].

Head normal forms

- A term *M* has a hnf if it reduces to a hnf.

- **Definition:** *H* and *H'* are **similar head normal forms** iff

  $H = \lambda x_1 x_2 \cdots x_m . x M_1 M_2 \cdots M_n$

  $H' = \lambda x_1 x_2 \cdots x_m . x M'_1 M'_2 \cdots M'_n$

  (same external structure)

- **Examples:**

  $\lambda xy.x(\Delta\Delta)x$ and $\lambda xy.xx(\Delta\Delta)$ are similar hnfs.

  $xy(\Delta\Delta)x$ and $xxy(\Delta\Delta)$ are similar hnfs.

  $\lambda xy.x(\Delta\Delta)$ and $\lambda xy.y(\Delta\Delta)$ are not similar.

Head normal forms

- **Lemma 1:** If $M \xrightarrow{*} H$ in hnf and $M \xrightarrow{*} H'$ in hnf, then H and H' are similar.
- **Lemma 2:** If M has a hnf, it has a minimum hnf H_0 such:

  for every hnf H, we have $M \xrightarrow[\text{head}]{*} H_0 \xrightarrow{*} H$.

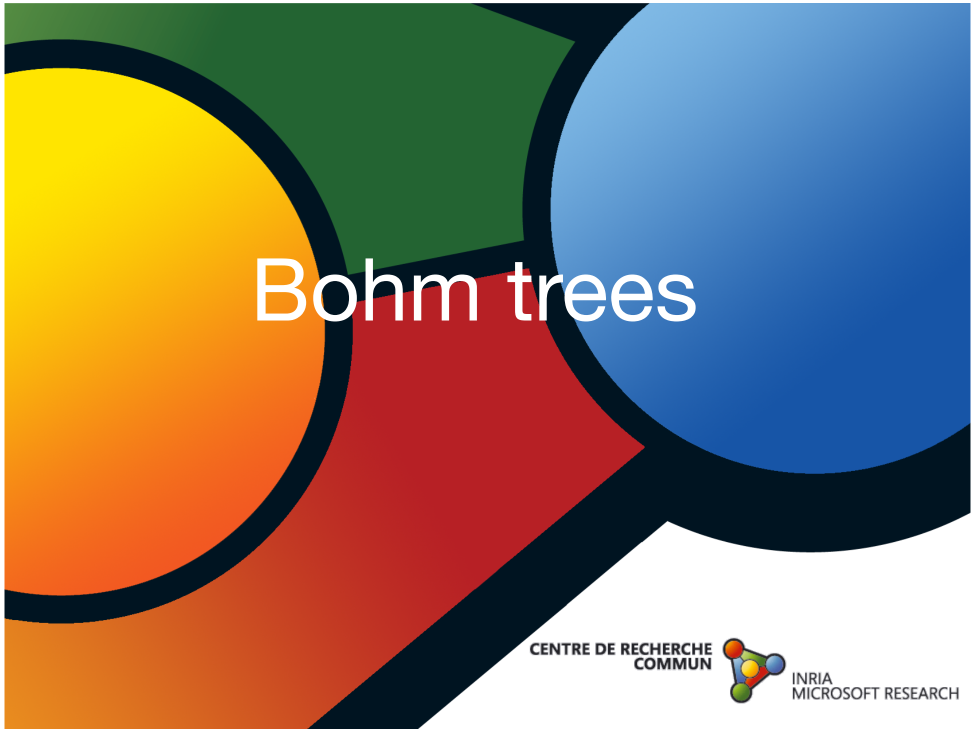
  where $\xrightarrow[\text{head}]{*}$ is head reduction.

  Proofs: easy.

- **Lemma 3:** If M has a hnf, then M is not totally undefined.

  Proof: easy again.
  Let $M \xrightarrow{*} \lambda x_1 x_2 \cdots x_m . x M_1 M_2 \cdots M_n$. We may suppose x bound. If not, we add an extra binder. So let $x = x_i$. Consider N_1, N_2, $\ldots N_m$ be any term, but $N_i = \lambda x_1 x_2 \cdots x_n . y$. Then $M N_1 N_2 \cdots N_n \xrightarrow{*} y$ in normal form, but $\Delta\Delta N_1 N_2 \cdots N_n$ has no normal form.

- We will later prove the opposite direction.



Exercices

1- Find Bohm context for xab and xac; for $\lambda xy.x$ and $\lambda xy.y$; for $x(xab)c$ and $x(xad)c$.

2- Bohm theorem can be generalized to n normal forms, pairwise distinct. Find Bohm context for xab, xac, and xbc.

3- Give examples of terms without hnf

4- Give examples of terms with hnf, but without normal forms

5- Prove that any normal form is also a head normal form

6- Show that Y has a hnf.

Bohm trees

- head normal forms are first level of the normal form of M

$$M \xrightarrow{*} \lambda x_1 x_2 \cdots x_m . x M_1 M_2 \cdots M_n.$$

- but we can iterate within M_1, M_2, $\ldots M_n$ and get second level

$$\begin{array}{l}
M_1 \xrightarrow{*} \lambda y_1 y_2 \cdots y_p . y N_1 N_2 \cdots N_q \\
M_2 \xrightarrow{*} \lambda z_1 z_2 \cdots z_r . z P_1 P_2 \cdots P_s \\
\vdots \\
M_n \xrightarrow{*} \lambda v_1 v_2 \cdots v_t . v Q_1 Q_2 \cdots Q_u
\end{array}$$

- and so on ...

Bohm trees

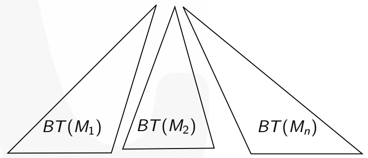
- this process gives the following tree-structure:

If M has no hnf

$$BT(M) = \Omega$$

If $M \xrightarrow{*} \lambda x_1 x_2 \cdots x_m.x\ M_1\ M_2\ \cdots\ M_n$

$$BT(M) = \lambda x_1 x_2 \cdots x_m.x$$



Need to define Bohm trees properly !

Bohm trees

$$BT(\Delta\Delta) = \Omega$$

$$BT(Ix(Ix)(Ix)) = \begin{array}{ccc} & x & \\ / & & \backslash \\ x & & x \end{array}$$

$$BT(Ix(\Delta\Delta)(Ix)) = \begin{array}{ccc} & x & \\ / & & \backslash \\ \Omega & & x \end{array}$$

$$BT(Ix(Ix)(\Delta\Delta)) = \begin{array}{ccc} & x & \\ / & & \backslash \\ x & & \Omega \end{array}$$

$$BT(Y) = \begin{array}{c} \lambda f.f \\ | \\ f \\ | \\ f \\ | \\ \vdots \end{array} = BT(Y')$$

$$Y = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$$

$$Y' = (\lambda xy.y(xxy))(\lambda xy.y(xxy))$$

Finite Bohm trees

- A **finite approximant** is any member of the following set of terms:

$$\begin{array}{rcll} a, b & ::= & \Omega & \\ & | & \lambda x_1 x_2 \cdots x_m.x a_1 a_2 \cdots a_n & (m \geq 0, n \geq 0) \end{array}$$

- examples of finite approximants:

  $x\Omega\Omega$

  $xx\Omega$

  $x\Omega x$

  $\lambda xy.xy(x\Omega)$

  $\lambda xy.x(\lambda z.y\Omega)$

- we call $\mathcal{N}$ the set of finite approximants

Finite Bohm trees

- Finite approximants can be ordered by following **prefix ordering**:

$$\Omega \leq a$$

$$a_1 \leq b_1,\ a_2 \leq b_2,\ \ldots\ a_n \leq b_n \text{ implies}$$

$$\lambda x_1 x_2 \cdots x_m . x a_1 a_2 \cdots a_n \leq \lambda x_1 x_2 \cdots x_m . x b_1 b_2 \cdots b_n$$

- examples:

$$x\Omega\Omega \leq xx\Omega$$

$$x\Omega\Omega \leq x\Omega x$$

$$\lambda xy.x\Omega \leq \lambda xy.xy$$

- thus $a \leq b$ iff several Ω's in a are replaced by finite approximants in b.

Finite Bohm trees

- **Lemma 1:**

  $\omega(M) = \Omega$ iff M is not in hnf.

  $\omega(\lambda x_1 x_2 \cdots x_m . x M_1 M_2 \cdots M_n) = \lambda x_1 x_2 \cdots x_m . x(\omega(M_1))(\omega(M_2)) \cdots (\omega(M_n))$

- **Lemma 2:** $M \longrightarrow N$ implies $\omega(M) \leq \omega(N)$

- **Lemma 3:** The set $\mathcal{N}$ of finite approximants is a conditional lattice with $\leq$.

- **Definition:** The set $\mathcal{A}(M)$ of direct approximants of M is defined as:

$$\mathcal{A}(M) = \{\omega(N) \mid M \xrightarrow{*} N\}$$

- **Lemma 4:** The set $\mathcal{A}(M)$ is a sublattice of $\mathcal{N}$ with same lub and glb.

  Proof: easy application of Church-Rosser + standardization.

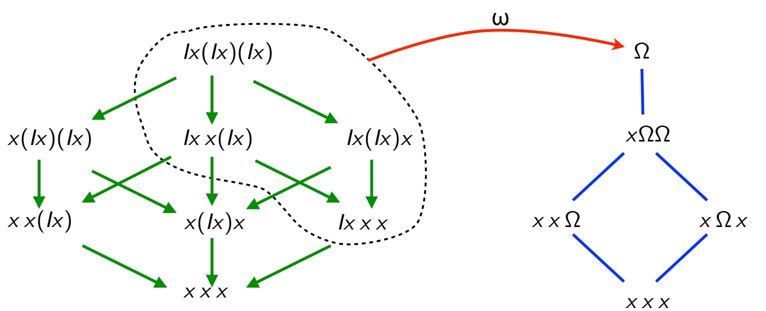
Finite Bohm trees

- ω*(M)* is **direct approximation of** *M*. It is obtained by replacing all redexes in *M* by constant Ω and applying exhaustively the two Ω-rules:

$$\Omega M \longrightarrow \Omega$$

$$\lambda x.\Omega \longrightarrow \Omega$$

- examples of direct approximation:



λ-terms

finite approximants

Bohm trees

- **Definition:** The Bohm tree of M is the set of prefixes of its direct approximants:

$$\mathrm{BT}(M) = \{a \in \mathcal{N} \mid a \leq b,\ b \in \mathcal{A}(M)\}$$

- In the terminology of partial orders and lattices, Bohm trees are ideals. Meaning they are directed sets and closed downwards. Namely:

  directed sets: $\forall a, b \in \mathrm{BT}(M),\ \exists c \in \mathrm{BT}(M),\quad a \leq c \ \wedge\ b \leq c.$

  ideals: $\forall b \in \mathrm{BT}(M),\ \forall a \in \mathcal{N},\ a \leq b \ \Rightarrow\ a \in \mathrm{BT}(M).$

- In fact, we made a completion by ideals. Take $\overline{\mathcal{N}} = \{A \mid A \subset \mathcal{N},\ A$ is an ideal$\}$

  Then $\langle \mathcal{N}, \leq \rangle$ can be completed as $\langle \overline{\mathcal{N}}, \subset \rangle$.

- Thus Bohm trees may be infinite and they are defined by the set of all their finite prefixes.

Bohm trees

- **Examples:**

  1- $\mathrm{BT}(\Delta\Delta) = \{\Omega\} = \mathrm{BT}(\Delta\Delta\Delta) = \mathrm{BT}(\Delta\Delta M)$

  2- $\mathrm{BT}((\lambda x.xxx)(\lambda x.xxx)) = \mathrm{BT}(YK) = \{\Omega\}$

  3- $\mathrm{BT}(M) = \{\Omega\}$ if M has no hnf

  4- $\mathrm{BT}(I) = \{\Omega, I\}$

  5- $\mathrm{BT}(K) = \{\Omega, K\}$

  6- $\mathrm{BT}(Ix(Ix)(Ix)) = \{\Omega, x\Omega\Omega, xx\Omega, x\Omega x, xxx\}$

  7- $\mathrm{BT}(Y) = \{\Omega, \lambda f.f\Omega, \lambda f.f(f\Omega), \ldots \lambda f.f^n(\Omega), \ldots\}$

  8- $\mathrm{BT}(Y') = \{\Omega, \lambda f.f\Omega, \lambda f.f(f\Omega), \ldots \lambda f.f^n(\Omega), \ldots\}$

Bohm tree semantics

- **Definition 1:** let the Bohm tree semantics be defined by:

$$M \equiv_{\mathrm{BT}} N \ \text{ iff } \ \mathrm{BT}(M) = \mathrm{BT}(N)$$

- **Definition 2:** we also consider Bohm tree ordering defined by:

$$M \sqsubseteq_{\mathrm{BT}} N \ \text{ iff } \ \mathrm{BT}(M) \subset \mathrm{BT}(N)$$

When clear from context, we just write $\equiv$ for $\equiv_{\mathrm{BT}}$ and $\sqsubseteq$ for $\sqsubseteq_{\mathrm{BT}}$.

- **New goal:** is Bohm tree semantics a (consistent) λ-theory ?

- We want to show that:

$$M \xrightarrow{*} N \ \text{ implies } \ M \equiv N$$

$$M \sqsubseteq N \ \text{ implies } \ C[M] \sqsubseteq C[N]$$

Bohm tree semantics

- **Proposition 1:** $M \xrightarrow{*} N$ implies $M \equiv N$

Proof: First $\mathrm{BT}(N) \subset \mathrm{BT}(M)$, since any approximant of N is one of M.
Conversely, take a in $\mathrm{BT}(M)$. We have $a \leq b = \omega(M')$ where $M \xrightarrow{*} M'$.
By Church-Rosser, there is N' such that $M' \xrightarrow{*} N'$ and $N \xrightarrow{*} N'$. By lemma 1, we have $\omega(M') \leq \omega(N')$.
Therefore $a \leq \omega(N')$ and $a \in \mathrm{BT}(N)$.

Homeworks

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Exercices

1- What is the finest (consistent) λ-theory.

2- Do carefully examples at slide just before Bohm tree semantics.

3- Give 2 λ-terms without normal form, but with distinct finite Bohm trees

4- Give 2 λ-terms with distinct infinite Bohm trees

5- Jacopini proved that $I \equiv \Delta\Delta$ makes a consistent theory. Why this is not contradictory with other results in this lecture?

6- Easy terms are terms which can be consistently equated to any other term. $\Delta\Delta$ is easy. Why again this is not contradictory with current chapter?