Plan

- Normalization
- Strong normalization
- Standardization theorem
- Normalization strategies
Reminders

• Redexes may be tracked with residuals

• One can define parallel reduction $\rightarrow^{\mathcal{F}}$ of a given set $\mathcal{F}$ of redexes by considering any of its finite developments.

• Lemma of parallel moves (other version of confluency lemma 1111)

• Cube lemma (consistency of residual relation w.r.t. finite developments)

• The labeled calculus was a technical tool to name redexes and prove Curry’s Finite Development Theorem.
Termination
Strong Normalization

- \( M \) is strongly normalizable iff every reduction from \( M \) is finite

\[
M \\
N \text{ normal form}
\]

- **Exercice:** which of following terms is strongly normalizable?

\[
l, II, \Delta \Delta, \Delta l, Y, Yl, YK, KL(\Delta \Delta)
\]

where \( l = \lambda x.x, \Delta = \lambda x.xx, K = \lambda x.\lambda y.x \) and \( Y = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx)) \).
Strong Normalization

• In typed lambda-calculi, all terms are strongly normalizable:
  • in 1st-order typed calculus, in system $F$, $F$-omega, terms are in $\mathcal{SN}$
  • terms of Coq are also strongly normalizable.

$\mathcal{SN} + \text{confluency}$  $\Rightarrow$  type-free $\lambda$-calculus

typed $\lambda$-terms $\leftrightarrow$ unique normal forms
Non termination

• In a fully expressive language, you have non-termination:

• in PCF + Y operator, in Ocaml, in Haskell, some terms are not in $SN$

• Confluency ensures deterministic calculations

• but possibly not terminating with a normal form.
Normalization

- $M$ is **normalizable** iff a reduction from $M$ leads to a normal form.

**Exercice:** Which of the following terms is normalizable?

$I, II, \Delta \Delta, \Delta I, Y, YI, YK, KI(\Delta \Delta)$

where $I = \lambda x.x$, $\Delta = \lambda x.xx$, $K = \lambda x.\lambda y.x$

and $Y = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$.
Normalization strategies

- Suppose $M$ is normalizable. Is there a strategy to reach the normal form? (normalizing strategy)

- Conversely, if $M$ has an infinite reduction, is there a strategy to fall in an infinite reduction? (perpetual strategies) [see Barendregt + Klop]

• Take: $M = (\lambda x.y)(\Delta\Delta) \xrightarrow{*} y$
  but $(\lambda x.y)(\Delta\Delta) \rightarrow (\lambda x.y)(\Delta\Delta) \rightarrow \cdots$

• Take: $M = I(\Delta(KI(\Delta\Delta))) \xrightarrow{*} I$
  but $M = I(\Delta(KI(\Delta\Delta))) \rightarrow I(\Delta(KI(\Delta\Delta))) \rightarrow \cdots$

• Take: $M = I(\Delta(K(\Delta\Delta)I)) \xrightarrow{*} \Delta\Delta \rightarrow \Delta\Delta \rightarrow \cdots$
  but $M \xrightarrow{*} N$ in normal form ??
Normalization strategies

• Take: $M = Y'(KI) \rightarrow I$

but $M = Y'(KI) \rightarrow KI(Y'(KI)) \rightarrow KI(KI(Y'(KI))) \rightarrow \cdots$

where $Y' = (\lambda xy.y(xx))(\lambda xy.y(xx))$

• Comparable to evaluation strategies in programming languages:

```java
static int f (int x, int y) {
  if (x == 0)
    return 1;
  else
    return f (x-1, f(x, y));
}
```

what is value of $f(1, 0)$? ???

• In PCF, it would be:

$$\ Y(\lambda fxy.\text{if}z \ x \ \text{then} \ 1 \ \text{else} \ f(x - 1)(f \ x \ y)) \ 1 \ 0$$
Normalization strategies

- In programming languages, evaluation strategies could be:
  - **call-by-value**: compute value of arguments of functions and pass values to the function parameters (Ocaml, Java)
  - **call-by-name**: pass symbolic expression of arguments to the function parameters and calculate them when needed.
  - **call-by-need**: variation of call-by-name in order to avoid recalculations of arguments (lazy languages -- Haskell)

- there are also CBV, CBN strategies in the lambda-calculus (we don’t do it here)

- Call-by-need is more complex [J JL’78, Lamping’90, Gonthier-Abadi-JJL’92]
Standardization
Standard reduction

Redex $R$ is to the left of redex $S$ if the $\lambda$ of $R$ is to the left of the $\lambda$ of $S$.

$$M = \cdots (\lambda x. A) B \cdots (\lambda y. C) D \cdots$$

or

$$M = \cdots (\lambda x. \cdots (\lambda y. C) D \cdots) B \cdots$$

or

$$M = \cdots (\lambda x. A)(\cdots (\lambda y. C) D \cdots) \cdots$$

The reduction $M = M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} M_2 \cdots \xrightarrow{R_n} M_n = N$ is standard iff for all $i, j$ ($0 < i < j \leq n$), redex $R_j$ is not a residual of redex $R'_j$ to the left of $R_i$ in $M_{i-1}$. 
Standard reduction

\[ M = (\lambda x.x)((\lambda f.3)(\lambda x.x)) \]

\[ (\lambda f.3)(\lambda x.x)((\lambda f.3)(\lambda x.x)) \]

\[ (\lambda f.3)(\lambda x.x)((\lambda x.x)3) \]

\[ (\lambda x.x)((\lambda x.x)3) \]

\[ (\lambda x.x)3((\lambda f.3)(\lambda x.x)) \]

\[ 3((\lambda f.3)(\lambda x.x)) \]

\[ (\lambda x.x)33 \]

\[ (\lambda x.x)3 \]

\[ N = 3((\lambda x.x)3) \]

\[ (\lambda x.xx)3 \]

\[ 33 \]
Standardization

- **Theorem [standardization] (Curry)**  Any reduction can be standardized.

- The **normal reduction** (each step contracts the leftmost-outermost redex) is a standard reduction.

- **Corollary [normalization]**  If $M$ has a normal form, the normal reduction reaches the normal form.
Standardization lemma

- **Notation**: write $R <_\ell S$ if redex $R$ is to the left of redex $S$.

- **Lemma 1** Let $R, S$ be redexes in $M$ such that $R <_\ell S$. Let $M \xrightarrow{S} N$. Then $R/S = \{R'\}$. Furthermore, if $T' <_\ell R'$, then $\exists T, T <_\ell R, T' \in T/S$. [one cannot create a redex through another more-to-the-left]

- **Proof of standardization thm**: [Klop] application of the finite developments theorem and previous lemma.
Standardization axioms

- 3 axioms are sufficient to get lemma 1

- **Axiom 1 [linearity]** \( S \not\leq \ell R \) implies \( \exists! R', R' \in R/S \)

- **Axiom 2 [context-freeness]** \( S \not\leq \ell R \) and \( R' \in R/S \) and \( T' \in T/S \) implies \( T \nRightarrow R \) iff \( T' \nRightarrow R' \) where \( \nRightarrow \) is \( <_\ell \) or \( >_\ell \)

- **Axiom 3 [left barrier creation]**
  
  \((R <_\ell S \) and \( \nexists T', T \in T'/S) \) implies \( R' <_\ell T \) where \( R/S = \{R'\} \)
Standardization proof

• Proof:

Each square is an application of the lemma of parallel moves. Let $\rho_i$ be the horizontal reductions and $\sigma_j$ the vertical ones. Each horizontal step is a parallel step, vertical steps are either elementary or empty.

We start with reduction $\rho_0$ from $M$ to $N$. Let $R_1$ be the leftmost redex in $M$ with residual contracted in $\rho_0$. By lemma 1, it has a single residual $R'_1$ in $M_1$, $M_2$, ... until it belongs to some $F_k$. Here $R'_1 \in F_2$. There are no more residuals of $R_1$ in $M_{k+1}$, $M_{k+2}$, ... .

Let $R_2$ be leftmost redex in $P_1$ with residual contracted in $\rho_1$. Here the unique residual is contracted at step $n$. Again with $R_3$ leftmost with residual contracted in $\rho_2$. Etc.
Standardization proof

• Proof (cont’d):

Now reduction $\sigma_0$ starting from $M$ cannot be infinite and stops for some $p$. If not, there is a rightmost column $\sigma_k$ with infinitely non-empty steps. After a while, this reduction is a reduction relative to a set $F^j_i$, which cannot be infinite by the Finite Development theorem.

Then $\rho_p$ is an empty reduction and therefore the final term of $\sigma_0$ is $N$. 
• **Proof (cont’d):**

We claim $\sigma_0$ is a standard reduction. Suppose $R_k$ ($k > i$) is residual of $S_i$ to the left of $R_i$ in $P_{i-1}$.

By construction $R_k$ has residual $S^j_k$ along $\rho_{i-1}$ contracted at some $j$ step. So $S^j_k$ is residual of $S_i$.

By the cube lemma, it is also residual of some $S^j_i$ along $\sigma_{j-1}$. Therefore there is $S^j_i$ in $F^j_i$ residual of $S_i$ leftmore or outer than $R_i$.

Contradiction.
Homeworks
Exercices

1- Show that $\Delta\Delta(II)$ has no normal form when $I = \lambda x.x$ and $\Delta = \lambda x.xx$.

2- Show that $\Delta\Delta M_1 M_2 \cdots M_n$ has no normal form for any $M_1, M_2, \ldots M_n$ ($n \geq 0$).

3- Show there is no $M$ whose reduction graph is exactly the following:

```
M
 /\ \\
/   \ \\
M_1 -> M_2 <- M_3 \\
 |   / \\
|  /   \\
| /     \\
|/      \\
N
```

4- Show that rightmost-outermost reduction may miss normal forms.

5- Show that if $M \xrightarrow{*} \lambda x.N$, there is a minimal $N_0$ such that for all $P$, such that if $M \xrightarrow{*} \lambda x.P$, then $N_0 \xrightarrow{*} P$. 