Reminders

- Redexes may be tracked with residuals.
- One can define parallel reduction $\rightarrow_p$ of a given set $\mathcal{F}$ of redexes by considering any of its finite developments.
- Lemma of parallel moves (other version of confluence lemma 1111)
- Cube lemma (consistency of residual relation w.r.t. finite developments)
- The labeled calculus was a technical tool to name redexes and prove Curry’s Finite Development Theorem.

Plan

- Normalization
- Strong normalization
- Standardization theorem
- Normalization strategies

Termination
Strong Normalization

- $M$ is strongly normalizable iff every reduction from $M$ is finite

Exercice: which of following terms is strongly normalizable?

$I, II, \Delta\Delta, \Delta I, Y, YI, YK, KI(\Delta\Delta)$

where $I = \lambda x.x$, $\Delta = \lambda x.xx$, $K = \lambda x.\lambda y.x$ and $Y = \lambda f.(\lambda x. f(xx))(\lambda x. f(xx))$.

Strong Normalization

- In typed lambda-calculi, all terms are strongly normalizable:
  - in 1st-order typed calculus, in system $F$, $F$-omega, terms are in $SN$.
  - terms of Coq are also strongly normalizable.

$SN +$ confluence $\rightarrow$ type-free $\lambda$-calculus

typed $\lambda$-terms $\leftrightarrow$ unique normal forms

Non termination

- In a fully expressive language, you have non-termination:
  - in PCF + Y operator, in Ocaml, in Haskell, some terms are not in $SN$.
  - Confluency ensures deterministic calculations.
  - but possibly not terminating with a normal form.

Normalization

- $M$ is normalizable iff a reduction from $M$ leads to a normal form.

Exercice: which of following terms is normalizable?

$I, II, \Delta\Delta, \Delta I, Y, YI, YK, KI(\Delta\Delta)$

where $I = \lambda x.x$, $\Delta = \lambda x.xx$, $K = \lambda x.\lambda y.x$ and $Y = \lambda f.(\lambda x. f(xx))(\lambda x. f(xx))$.  

$SN$ $\rightarrow$ normal form

$\rightarrow$ infinite reduction

but normal form
Normalization strategies

- Suppose $M$ is normalizable. Is there a strategy to reach the normal form? (normalizing strategy)
- Conversely, if $M$ has an infinite reduction, is there a strategy to fall in an infinite reduction? (perpetual strategies) [see Barendregt + Klop]
- Take: $M = (\lambda x. y)(\Delta \Delta) \rightsquigarrow y$
  but $(\lambda x. y)(\Delta \Delta) \rightarrow (\lambda x. y)(\Delta \Delta) \rightarrow \cdots$
- Take: $M = I(\Delta(KI(\Delta \Delta))) \rightsquigarrow I$
  but $M = I(\Delta(KI(\Delta \Delta))) \rightarrow I(\Delta(KI(\Delta \Delta))) \rightarrow \cdots$
- Take: $M = I(\Delta(K(\Delta \Delta))) \rightarrow \Delta \Delta \rightarrow \Delta \Delta \rightarrow \cdots$
  but $M \rightsquigarrow N$ in normal form??
**Standard reduction**

Redex \( R \) is to the left of redex \( S \) if the \( \lambda \) of \( R \) is to the left of the \( \lambda \) of \( S \).

\[
M = \cdots (\lambda x.A)B \cdots (\lambda y.C)D \cdots \underbrace{R}_{S}
\]

or

\[
M = \cdots (\lambda x \cdots (\lambda y.C)D \cdots )B \cdots \underbrace{R}_{S}
\]

or

\[
M = \cdots (\lambda x.A)(\cdots (\lambda y.C)D \cdots ) \cdots \underbrace{R}_{S}
\]

The reduction \( M = M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} \cdots \xrightarrow{R_n} M_n = N \) is standard iff for all \( i, j \) (\( 0 < i < j \leq n \)), redex \( R_j \) is not a residual of redex \( R'_j \) to the left of \( R_i \) in \( M_{i-1} \).

**Standardization**

- **Theorem [standardization] (Curry)** Any reduction can be standardized.

- **The normal reduction** (each step contracts the leftmost-outermost redex) is a standard reduction.

- **Corollary [normalization]** If \( M \) has a normal form, the normal reduction reaches the normal form.

**Standardization lemma**

- **Notation**: write \( R <_l S \) if redex \( R \) is to the left of redex \( S \).

- **Lemma 1** Let \( R, S \) be redexes in \( M \) such that \( R <_l S \). Let \( M \xrightarrow{S} N \).

Then \( R/S = \{R'\} \). Furthermore, if \( T' <_l R' \), then \( \exists T, T <_l R, T' \in T/S \).

(one cannot create a redex through another more-to-the-left)

- **Proof of standardization thm**: [Klop] application of the finite developments theorem and previous lemma.
Standardization axioms

- 3 axioms are sufficient to get lemma 1

- **Axiom 1 [linearity]** \( S \not\leq R \) implies \( \exists! R', R' \in R/S \)

- **Axiom 2 [context-freeness]** \( S \not\leq R \) and \( R' \in R/S \) and \( T' \in T/S \) implies \( T \not\leq R \) iff \( T' \not\leq R' \) where \( \not\leq \) is \( <_{\ell} \) or \( >_{\ell} \)

- **Axiom 3 [left barrier creation]**

  \( (R <_{\ell} S \) and \( T \in T'/S) \) implies \( R' <_{\ell} T \) where \( R/S = \{R'\} \)

Standardization proof

- **Proof (cont'd):**

Now reduction \( \sigma_0 \) starting from \( M \) cannot be infinite and stops for some \( p \). If not, there is a rightmost column \( \sigma_4 \) with infinitely non-empty steps. After a while, this reduction is a reduction relative to a set \( \mathcal{F}_1 \), which cannot be infinite by the Finite Development theorem.

Then \( \rho_p \) is an empty reduction and therefore the final term of \( \sigma_0 \) is \( N \).

Standardization proof

- **Proof:**

Each square is an application of the lemma of parallel moves. Let \( \rho_i \) be the horizontal reductions and \( \sigma_j \) the vertical ones. Each horizontal step is a parallel step, vertical steps are either elementary or empty.

We start with reduction \( \rho_0 \) from \( M \) to \( N \). Let \( R_1 \) be the leftmost redex in \( M \) with residual contracted in \( \rho_0 \). By lemma 1, it has a single residual \( R_1' \) in \( M_1, M_2, \ldots \) until it belongs to some \( \mathcal{F}_s \). Here \( R_1' \in \mathcal{F}_2 \). There are no more residuals of \( R_1 \) in \( M_{k+1}, M_{k+2}, \ldots \).

Let \( R_2 \) be leftmost redex in \( P_1 \) with residual contracted in \( \rho_1 \). Here the unique residual is contracted at step \( n \). Again with \( R_3 \) leftmost with residual contracted in \( \rho_2 \), etc.
Exercices

1- Show that $\Delta \Delta (I)$ has no normal form when $I = \lambda x.x$ and $\Delta = \lambda x.xx$.

2- Show that $\Delta \Delta M_1 M_2 \cdots M_n$ has no normal form for any $M_1, M_2, \ldots M_n$ ($n \geq 0$).

3- Show there is no $M$ whose reduction graph is exactly the following:

```
M
 \downarrow
M_1 \quad M_2 \quad M_3
 \downarrow \quad \downarrow \quad \downarrow
N
```

4- Show that rightmost-outermost reduction may miss normal forms.

5- Show that if $M \rightarrow \lambda x.N$, there is a minimal $N_0$ such that for all $P$, such that if $M \rightarrow \lambda x.P$, then $N_0 \rightarrow P$. 