Lambda-Calculus (III-1)

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Plan

- language
- abbreviations
- local confluency
- Church Rosser theorem
- Redexes and residuals

λ-calculus

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The lambda-calculus

- Lambda terms

M, N, P	::=	*x, y, z, ...*	(variables)
	|	*(λx.M)*	(*M* as function of *x*)
	|	*(M N)*	(*M* applied to *N*)
	|	*c, d, ...*	(*constants*)

- Calculations "reductions"

$$((\lambda x.M)N) \longrightarrow M\{x := N\}$$

Abbreviations

$MM_1M_2\cdots M_n$	for	$(\cdots((MM_1)M_2)\cdots M_n)$
$(\lambda x_1 x_2 \cdots x_n\,.\,M)$	for	$(\lambda x_1.(\lambda x_2.\cdots(\lambda x_n\,.\,M)\cdots))$

external parentheses and parentheses after a dot may be forgotten

Exercice 1

Write following terms in long notation:

$\lambda x.x,\ \lambda x.\lambda y.x,\ \lambda xy.x,\ \lambda xyz.y,\ \lambda xyz.zxy, \lambda xyz.z(xy),$
$(\lambda x.\lambda y.x)MN,\ (\lambda xy.x)MN,\ (\lambda xy.y)MN,\ (\lambda xy.y)(MN)$

Examples

$$(\lambda x.x)N \longrightarrow N$$

$$(\lambda f.f\ N)(\lambda x.x) \longrightarrow (\lambda x.x)N \longrightarrow N$$

$$(\lambda x.xx)(\lambda x.xN) \longrightarrow (\lambda x.xN)(\lambda x.xN) \longrightarrow (\lambda x.xN)N \longrightarrow NN$$

$$(\lambda x.xx)(\lambda x.xx) \longrightarrow (\lambda x.xx)(\lambda x.xx) \longrightarrow \cdots$$

$$Y_f = (\lambda x.f(xx))(\lambda x.f(xx)) \longrightarrow f((\lambda x.f(xx))(\lambda x.f(xx))) = f(Y_f)$$

$$f(Y_f) \longrightarrow f(f(Y_f)) \longrightarrow \cdots \longrightarrow f^n(Y_f) \longrightarrow \cdots$$

Substitution

$x\{y := P\} = x \qquad\qquad c\{y := P\} = c$

$y\{y := P\} = P$

$(MN)\{y := P\} = M\{y := P\}\ N\{y := P\}$

$(\lambda y.M)\{y := P\} = \lambda y.M$

$(\lambda x.M)\{y := P\} = \lambda x'.M\{x := x'\}\{y := P\}$

where $x' = x$ if y not free in M or x not free in P,
otherwise x' is the first variable not free in M and P.
(we suppose that the set of variables is infinite and enumerable)

Free variables

$\mathsf{var}(x) = \{x\} \qquad\qquad \mathsf{var}(c) = \emptyset$

$\mathsf{var}(MN) = \mathsf{var}(M) \cup \mathsf{var}(N)$

$\mathsf{var}(\lambda x.M) = \mathsf{var}(M) - \{x\}$

Conversion rules

$$
\begin{array}{rcll}
\lambda x.M & \longrightarrow_\alpha & \lambda x'.M\{x := x'\} & (x' \notin \mathrm{var}(M)) \\
(\lambda x.M)N & \longrightarrow_\beta & M\{x := N\} & \\
\lambda x.Mx & \longrightarrow_\eta & M & (x \notin \mathrm{var}(M))
\end{array}
$$

- left-hand-side of conversion rule is a **redex** (reductible expression)
- α-redex, β-redex, η-redex, ...
- we forget indices when clear from context, often β

Reduction step

- let R be a redex in M. Then one can contrat redex R in M and get N:

$$M \xrightarrow{R} N$$

Reductions

$$M \xrightarrow{\star} N \quad \text{when} \quad M = M_0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow \cdots M_n = N \quad (n \geq 0)$$

- same with explicit contracted redexes

$$M = M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} M_2 \cdots \xrightarrow{R_n} M_n = N$$

- and with named reductions

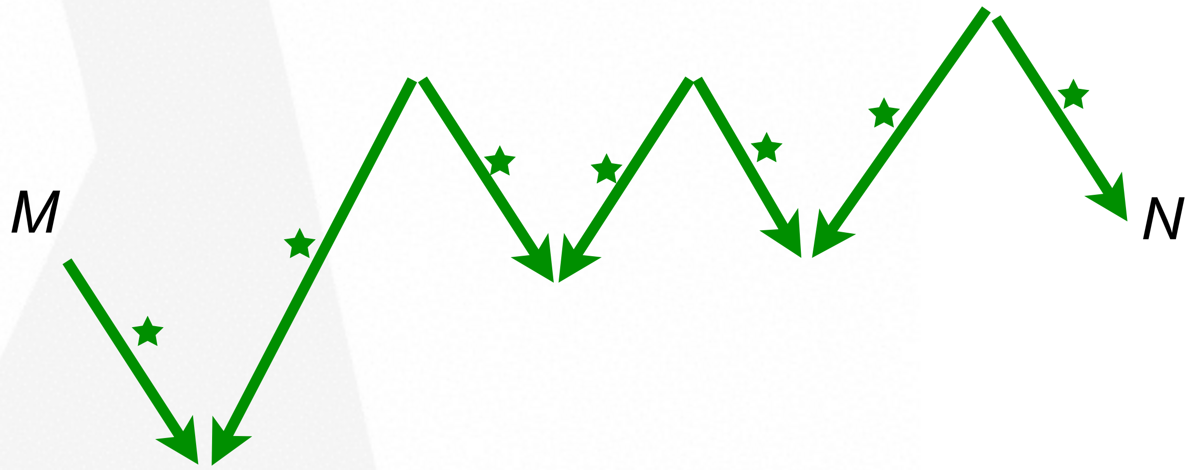
$$\rho : M = M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} M_2 \cdots \xrightarrow{R_n} M_n = N$$

- we speak of redex occurences when specifying reduction steps, but it is convenient to confuse redexes and redex occurences when clear from context

Lambda theories

$M =_\beta N$ when M and N are related by a zigzag of reductions

M and N are said **interconvertible**



- Also $M =_\alpha N$, $M =_\eta N$, $M =_{\beta,\eta} N$, ...
- Interconvertibility is symmetric, reflexive, transivite closure of reduction relation
- or with notations of mathematical logic:

$$\alpha \vdash M = N, \quad \beta \vdash M = N, \quad \eta \vdash M = N, \quad \beta + \eta \vdash M = N, \ldots$$

- the syntactic equality $M = N$ will often stand for $M =_\alpha N$.

Exercice 3

- Show that $M \longrightarrow N$ implies $\mathrm{var}(N) \subset \mathrm{var}(M)$.

- Find terms M such that:

$$M \longrightarrow M$$

$$M = M_0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow \cdots M_n = M \quad (M_i \text{ all distinct})$$

$$M =_\beta x\, M$$

$$M =_\beta \lambda x.M$$

$$M =_\beta MM$$

$$M =_\beta MN_1N_2 \cdots N_n \text{ for all } N_1, N_2, \ldots N_n$$

- Find term Y such that, for any M:

$$YM =_\beta M(YM)$$

- Find Y' such that, for any M:

$$Y'M \xrightarrow{\star} M(Y'M)$$

- (difficult) Show there is only one redex R such that $R \longrightarrow R$

Normal forms

- An expression M without redexes **is in** normal form

$$M \not\rightarrow$$

- If M reduces to a normal form, then M **has a** normal form

$$M \xrightarrow{*} N, \quad N \text{ in normal form}$$

Exercice 4

- which of following terms are in β-normal form ?

in $\beta\eta$-normal form ?

$\lambda x.x$

$\lambda xy.x$

$\lambda xy.xy$

$\lambda xy.x((\lambda x.y(xx))(\lambda x.y(xx)))$

$\lambda x.x(\lambda xy.x)(\lambda x.x)$

$\lambda xy.x(\lambda xy.x)(\lambda x.yx)$

$\lambda xy.x((\lambda x.xx)(\lambda x.xx))y$

Exercice 5

- Show that if M is in normal form and $M \xrightarrow{\star} N$, then $M = N$
- Show that:

1- $\lambda x.M \xrightarrow{\star} N$ implies $N = \lambda x.N'$ and $M \xrightarrow{\star} N'$

2- $MN \xrightarrow{\star} P$ implies $M \xrightarrow{\star} M'$, $N \xrightarrow{\star} N'$ and $P = M'N'$
or $M \xrightarrow{\star} \lambda x.M'$, $N \xrightarrow{\star} N'$ and $M'\{x := N'\} \xrightarrow{\star} P$

3- $xM_1M_2 \cdots M_n \xrightarrow{\star} N$ implies $M_1 \xrightarrow{\star} N_1$, $M_2 \xrightarrow{\star} N_2$, ... $M_n \xrightarrow{\star} N_n$
and $xN_1N_2 \cdots N_n = N$

4- $M\{x := N\} \xrightarrow{\star} \lambda y.P$ implies $M \xrightarrow{\star} \lambda y.M'$ and $M'\{x := N\} \xrightarrow{\star} P$
or $M \xrightarrow{\star} xM_1M_2 \cdots M_n$ and $NM_1\{x := N\} \cdots M_n\{x := N\} \xrightarrow{\star} \lambda y.P$

Reduction (axiomatic def.)

We can define reduction $\longrightarrow$ axiomatically by following axioms and rules:

- **Definition [beta reduction]:**

[App1 Rule] $\dfrac{M \longrightarrow M'}{MN \longrightarrow M'N}$ [App2 Rule] $\dfrac{N \longrightarrow N'}{MN \longrightarrow MN'}$

[Abs Rule] $\dfrac{M \longrightarrow M'}{\lambda x.M \longrightarrow \lambda x.M'}$ [//Beta Axiom] $(\lambda x.M)N \longrightarrow M\{x := N\}$

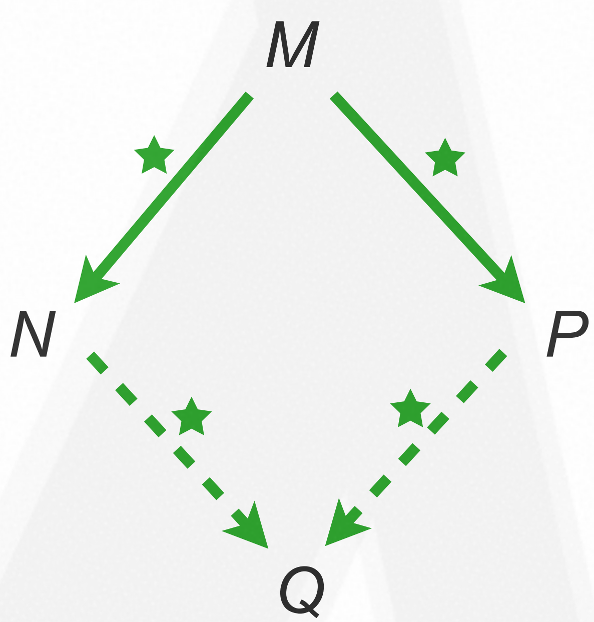
Exercice 6

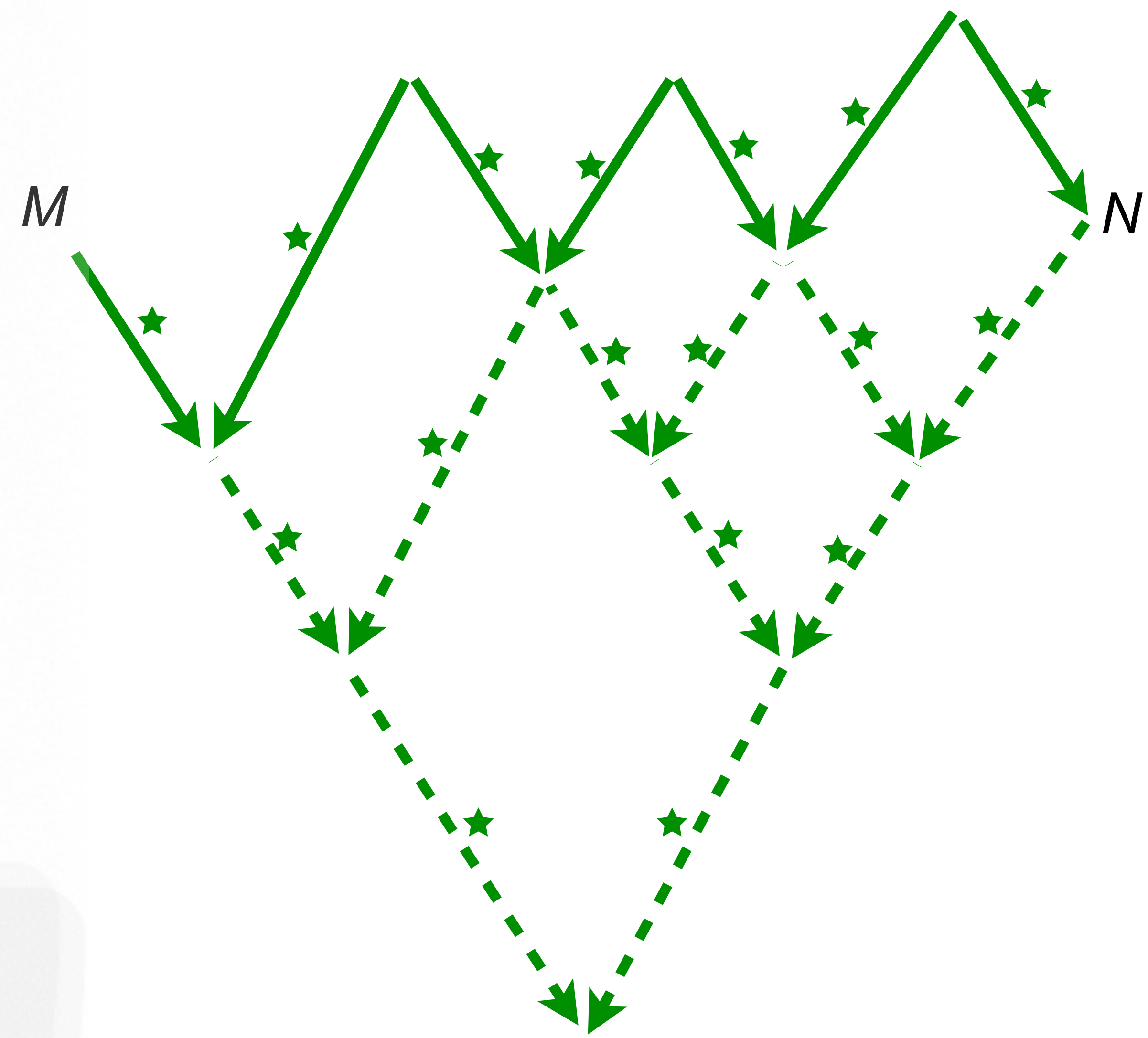
Give axiomatic definition for $\xrightarrow{\star}$, $\longrightarrow_{\eta}$, $\xrightarrow{\star}_{\beta,\eta}$.

Confluency

Confluency

Question: If $M \xrightarrow{*} N$ and $M \xrightarrow{*} P$, then $N \xrightarrow{*} Q$ and $P \xrightarrow{*} Q$ for some Q ?



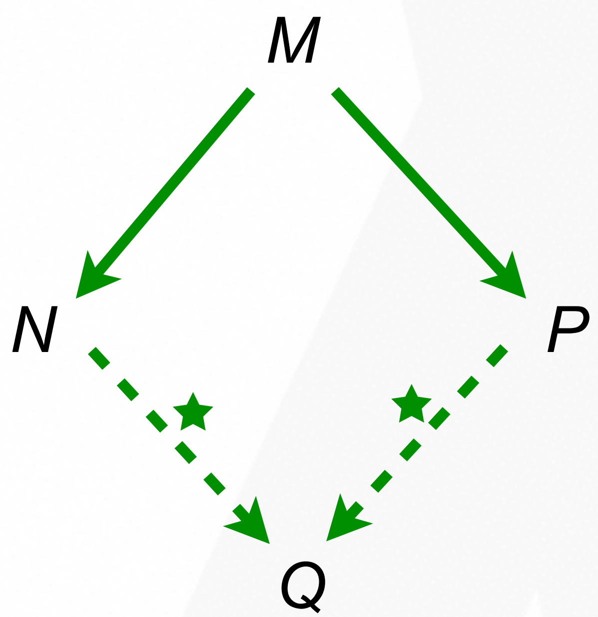


Corollary: [unicity of normal forms]

If $M \xrightarrow{*} N$ in normal form and $M \xrightarrow{*} N'$ in normal form, then $N = N'$.

Local confluency

- **Theorem 1 [`lemma 11**`]:** If $M \longrightarrow N$ and $M \longrightarrow P$, then $N \xrightarrow{\star} Q$ and $P \xrightarrow{\star} Q$ for some Q



- **Lemma 1:** $M \longrightarrow N$ implies $P\{x := M\} \xrightarrow{\star} P\{x := N\}$
- **Lemma 2:** $M \longrightarrow N$ implies $M\{x := P\} \longrightarrow N\{x := P\}$

- **Substitution lemma:** $M\{x := N\}\{y := P\} = M\{y := P\}\{x := N\{y := P\}\}$

  when x not free in P

Local confluency

- **Substitution lemma:** $M\{x := N\}\{y := P\} \;=\; M\{y := P\}\{x := N\{y := P\}\}$

  when x not free in P

Proof:

Write $A^* = A\{x := N\}\{y := P\}$ and $A^\dagger = A\{y := P\}\{x := N\{y := P\}\}$ for any A.

Case 1: $M = x$. Then $M^* = N\{y := P\} = M^\dagger$.

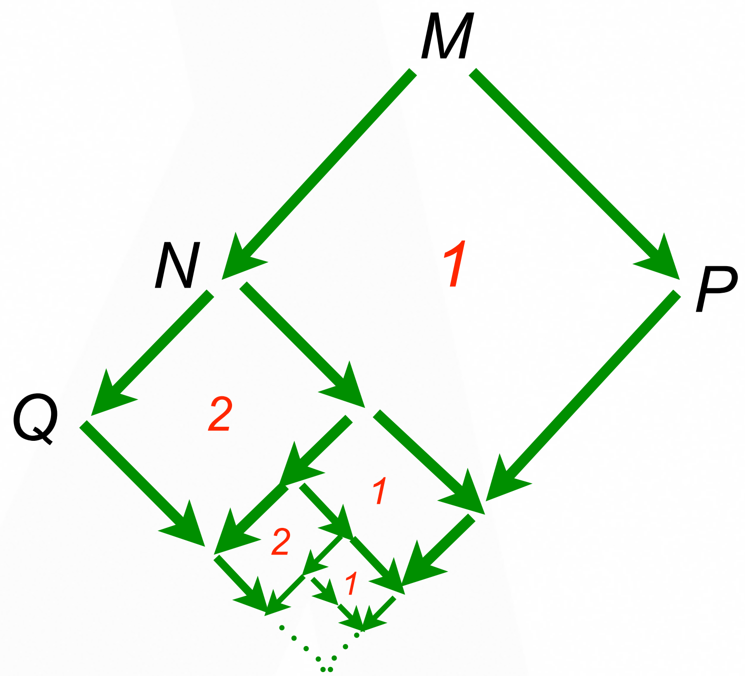
Case 2: $M = y$. Then $M^* = P = P\{x := A\}$ for any A since $x \notin \mathrm{var}(P)$. Therefore $M^* = P\{x := N\{y := P\}\} = M^\dagger$.

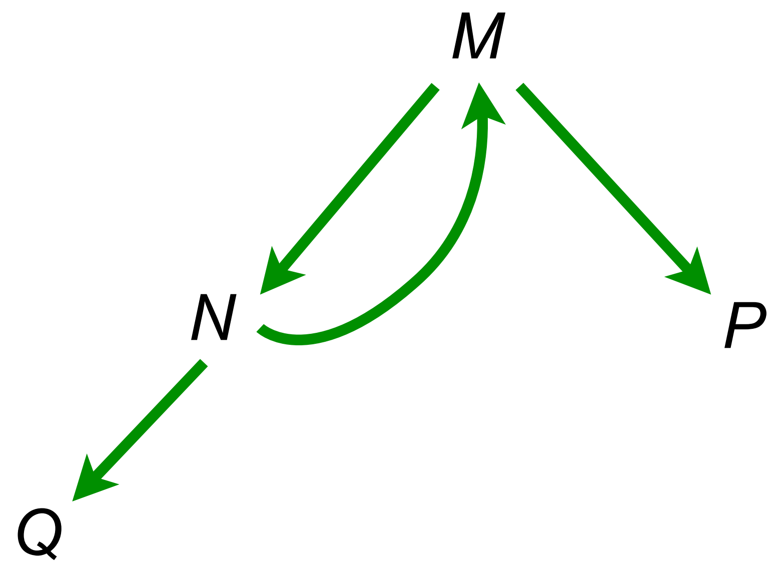
Case 3: $M = M_1 M_2$. This case is easy by induction.

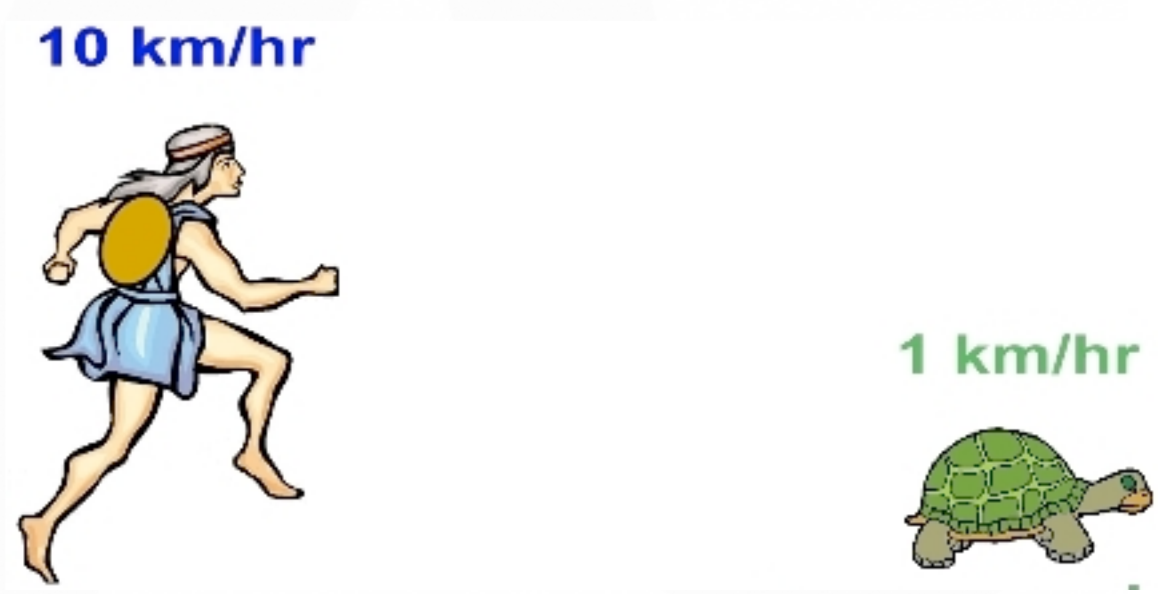
Case 4: $M = \lambda z.M_1$. We assume (by α-conversion) that z is a fresh variable neither in N, nor in P. Then induction is easy, since $M^* = \lambda z.M_1^*$ and $M^\dagger = \lambda z.M_1^\dagger$. This case is then similar to the previous one.

Confluency

- **Fact:** local confluency does not imply confluency







Confluency

We define $\twoheadrightarrow\!\!\!\!\!\!\!\!\not\,\not\;\;$ such that $\rightarrow \subset \;\not\!\not\!\rightarrow\; \subset \xrightarrow{*}$

- **Definition [parallel reduction]:**

$$\text{[Var Axiom]}\; x \;\not\!\not\!\rightarrow\; x \qquad\qquad \text{[Const Axiom]}\; c \;\not\!\not\!\rightarrow\; c$$

$$\text{[App Rule]}\;\frac{M \;\not\!\not\!\rightarrow\; M' \quad N \;\not\!\not\!\rightarrow\; N'}{MN \;\not\!\not\!\rightarrow\; M'N'} \qquad\qquad \text{[Abs Rule]}\;\frac{M \;\not\!\not\!\rightarrow\; M'}{\lambda x.M \;\not\!\not\!\rightarrow\; \lambda x.M'}$$

$$\text{[//Beta Rule]}\;\frac{M \;\not\!\not\!\rightarrow\; M' \quad N \;\not\!\not\!\rightarrow\; N'}{(\lambda x.M)N \;\not\!\not\!\rightarrow\; M'\{x := N'\}}$$

- Example:

$$\cfrac{\cfrac{x \;\not\!\not\!\rightarrow\; x \quad z \;\not\!\not\!\rightarrow\; z}{Iz \;\not\!\not\!\rightarrow\; z} \qquad \cfrac{x \;\not\!\not\!\rightarrow\; x \quad z \;\not\!\not\!\rightarrow\; z}{Iz \;\not\!\not\!\rightarrow\; z}}{Iz(Iz) \;\not\!\not\!\rightarrow\; zz}$$

$$I = \lambda x.x$$

Confluency

- Goal is to prove **strongly local confluency:**

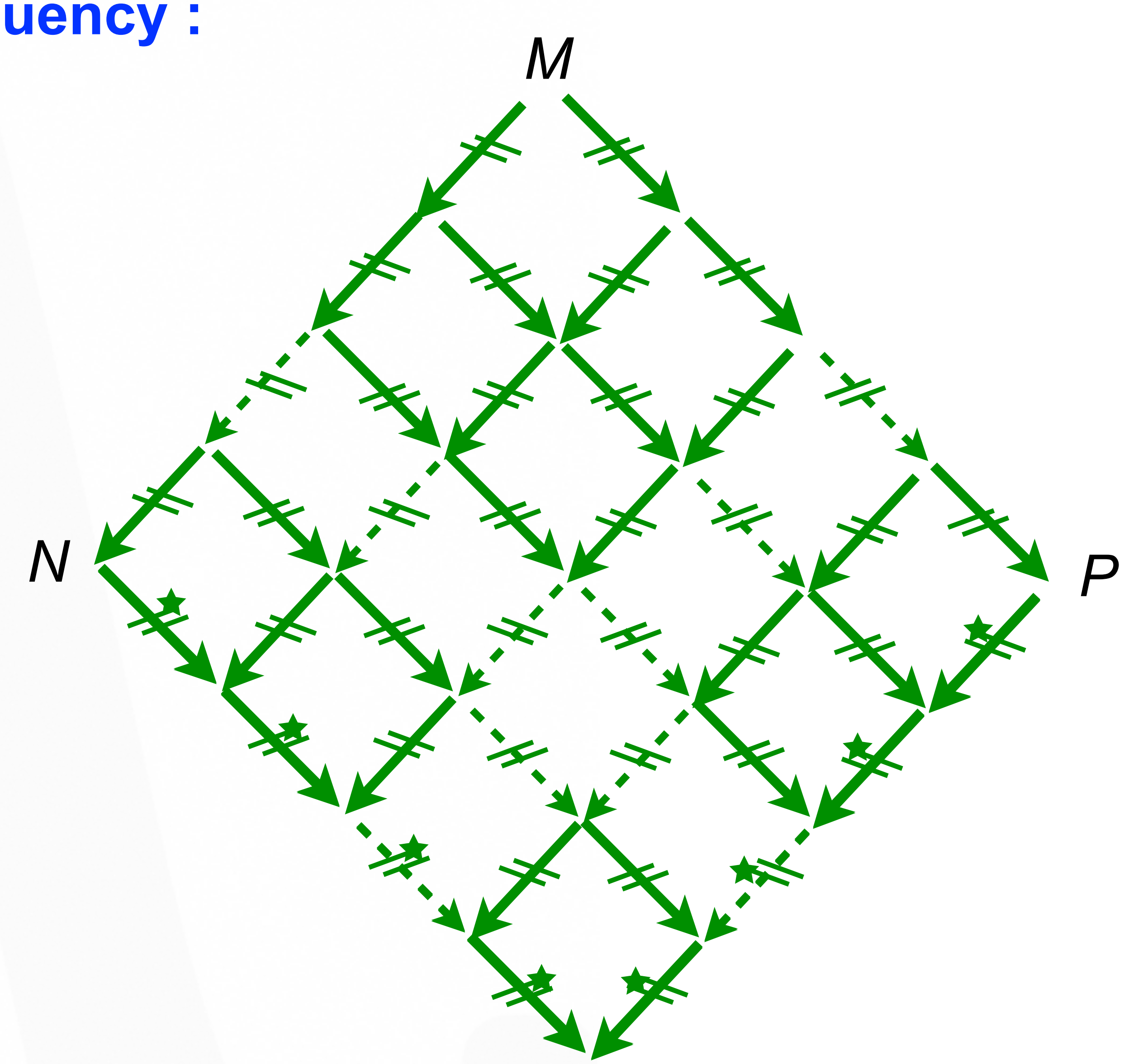
$$
\begin{tikzcd}
& M \arrow[dl, "/\!/"'] \arrow[dr, "/\!/"] & \\
N \arrow[dr, dashed, "/\!/"'] & & P \arrow[dl, dashed, "/\!/"] \\
& Q &
\end{tikzcd}
$$

- Example:

$$
\begin{tikzcd}
(\lambda x.xx)(Iz) \arrow[r, "/\!/"] \arrow[dr, "/\!/"'] & (\lambda x.xx)z \arrow[dr, "/\!/"] & \\
& Iz(Iz) \arrow[r, "/\!/"] & zz
\end{tikzcd}
$$

Confluency

- **Proof of confluency :**



Confluency

- **Lemma 4:** $M \mathrel{/\!\!/\!\!\!\!\!\rightarrow} N$ and $P \mathrel{/\!\!/\!\!\!\!\!\rightarrow} Q$ implies $M\{x := P\} \mathrel{/\!\!/\!\!\!\!\!\rightarrow} N\{x := Q\}$
- **Lemma 5:** If $M \mathrel{/\!\!/\!\!\!\!\!\rightarrow} N$ and $M \mathrel{/\!\!/\!\!\!\!\!\rightarrow} P$, then $N \mathrel{/\!\!/\!\!\!\!\!\rightarrow} Q$ and $N \mathrel{/\!\!/\!\!\!\!\!\rightarrow} Q$ for some Q.

  Proofs L6/L7: structural induction + substitution lemma.

- **Lemma 6:** If $M \longrightarrow N$, then $M \mathrel{/\!\!/\!\!\!\!\!\rightarrow} N$.
- **Lemma 7:** If $M \mathrel{/\!\!/\!\!\!\!\!\rightarrow} N$, then $M \xrightarrow{*} N$.

  Proofs L6/L7: obvious.

- **Theorem 2 [Church-Rosser]:**

  If $M \xrightarrow{*} N$ and $M \xrightarrow{*} P$, then $N \xrightarrow{*} Q$ and $P \xrightarrow{*} Q$ for some Q.

Confluency

- previous axiomatic method is due to **Tait** and **Martin-Löf**
- Tait--Martin-Löf's method models inside-out parallel reductions
- there are other proofs with explicit redexes

$\downarrow$

- Curry's finite developments

Residuals of redexes

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Residuals of redexes

- tracking redexes while contracting others
- examples:

$$\Delta = \lambda x.\, xx \quad I = \lambda x.x \quad K = \lambda xy.x$$

$$\Delta(\underbrace{Ia}) \longrightarrow \underbrace{Ia}(\underbrace{Ia})$$

$$\underbrace{Ia}(\Delta(Ib)) \longrightarrow \underbrace{Ia}(Ib(Ib))$$

$$\underbrace{I(\Delta(Ia))} \longrightarrow \underbrace{I(Ia(Ia))}$$

$$\underbrace{\Delta(Ia)} \longrightarrow Ia(Ia))$$

$$Ia(\Delta(\underbrace{Ib})) \longrightarrow Ia(\underbrace{Ib}(\underbrace{Ib}))$$

$$\underbrace{\Delta\Delta} \longrightarrow \Delta\Delta$$

$$(\lambda x.Ia)(\underbrace{Ib}) \longrightarrow Ia$$

Residuals of redexes

- when R is redex in M and $M \xrightarrow{S} N$

  the set R/S of **residuals** of R in N is defined by inspecting relative positions of R and S in M:

1- R and S disjoint, $M = \cdots \underbrace{R}_{} \cdots S \cdots \xrightarrow{S} \cdots \underbrace{R}_{} \cdots S' \cdots = N$

2- S in $R = (\lambda x.A)B$

2a- S in A, $M = \cdots \underbrace{(\lambda x. \cdots S \cdots)B}_{} \cdots \xrightarrow{S} \cdots \underbrace{(\lambda x. \cdots S' \cdots)B}_{} \cdots = N$

2b- S in B, $M = \cdots \underbrace{(\lambda x.A)(\cdots S \cdots)}_{} \cdots \xrightarrow{S} \cdots \underbrace{(\lambda x.A)(\cdots S' \cdots)}_{} \cdots = N$

3- R in $S = (\lambda y.C)D$

3a- R in C, $M = \cdots (\lambda y. \cdots \underbrace{R}_{} \cdots)D \cdots \xrightarrow{S} \cdots\cdots \underbrace{R\{y := D\}}_{} \cdots\cdots = N$

3b- R in D, $M = \cdots (\lambda y.C)(\cdots \underbrace{R}_{} \cdots) \cdots \xrightarrow{S} \cdots (\cdots \underbrace{R}_{} \cdots) \cdots (\cdots \underbrace{R}_{} \cdots) \cdots = N$

4- R is S, no residuals of R.

Residuals of redexes

- when ρ is a reduction from M to N, i.e. $\rho : M \xrightarrow{*} N$

  the set of residuals of R by ρ is defined by **transitivity** on the length of ρ

  and is written R/ρ

- notice that we can have $S \in R/\rho$ and $R \neq S$

  residuals may **not** be syntacticly **equal** (see previous 3rd example)

- residuals **depend on reductions**. Two reductions between same terms may produce two distinct sets of residuals.

- a redex is residual of a **single** redex (the inverse of the residual relation is a function): $R \in S/\rho$ and $R \in T/\rho$ implies $S = T$

Exercice 7

- Find redex R and reductions ρ and σ between M and N such that residuals of R by ρ and σ differ. Hint: consider $M = I(Ix)$
- Show that residuals of nested redexes keep nested.
- Show that residuals of disjoint redexes may be nested.
- Show that residuals of a redex may be nested after several reduction steps.

Created redexes

- A redex is **created by reduction** ρ if it is not a residual by ρ of a redex in initial term. Thus R is created by ρ when $\rho : M \xrightarrow{\star} N$ and $\nexists S,\ R \in S/\rho$

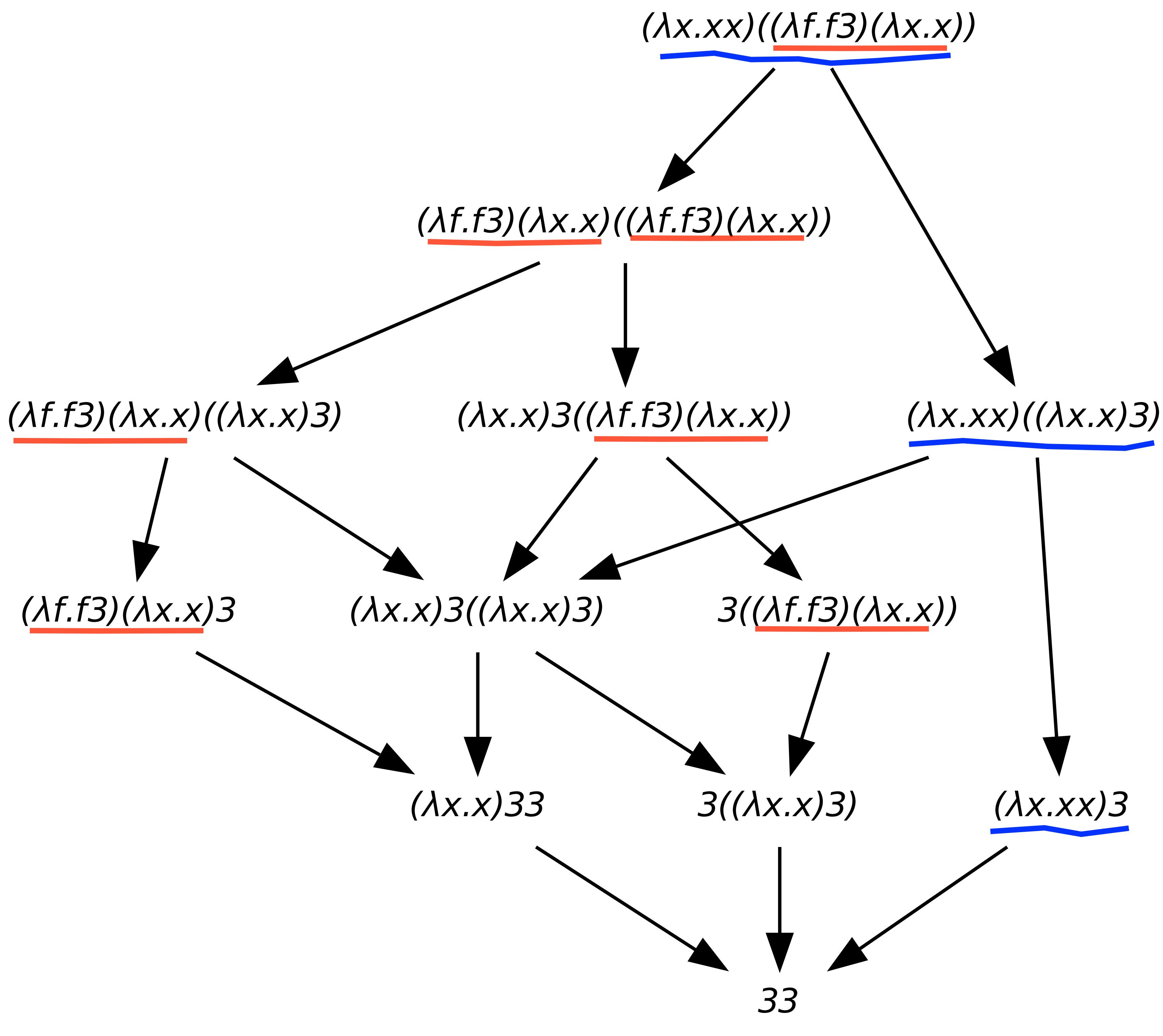
$$(\lambda x.xa)I \rightarrow \underbrace{Ia}$$

$$IIa \rightarrow \underbrace{Ia}$$

$$(\lambda xy.xy)ab \rightarrow \underbrace{(\lambda y.ay)b}$$

$$\Delta\Delta \rightarrow \underbrace{\Delta\Delta}$$

Residuals of redexes



Homeworks

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Exercices 8

- Show that:

1- $M \rightarrow_\eta N \rightarrow P$ implies $M \rightarrow Q \xrightarrow{*}_\eta P$ for some Q

2- $M \xrightarrow{*}_\eta N \xrightarrow{*} P$ implies $M \xrightarrow{*} Q \xrightarrow{*}_\eta P$ for some Q

3- $M \xrightarrow{*}_{\beta,\eta} N$ implies $M \xrightarrow{*} P \xrightarrow{*}_\eta N$ for some P

4- $M \rightarrow N$ and $M \rightarrow_\eta P$ implies $N \xrightarrow{*}_\eta Q$ and $P \xrightarrow{1} Q$ for some Q

5- $M \xrightarrow{*}_\eta N$ and $M \xrightarrow{*}_\eta P$ implies $N \xrightarrow{*}_\eta Q$ and $P \xrightarrow{*}_\eta Q$ for some Q

6- $M \xrightarrow{*}_{\beta,\eta} N$ and $M \xrightarrow{*}_{\beta,\eta} P$ implies $N \xrightarrow{*}_{\beta,\eta} Q$ and $P \xrightarrow{*}_{\beta,\eta} Q$ for some Q

Therefore $\xrightarrow{*}_{\beta,\eta}$ is confluent.

- Show same property for β-reduction and η-expansion $(\rightarrow \cup \leftarrow_\eta)^*$

Exercices

7- Show there is no M such that $M \xrightarrow{\star} Kac$ and $M \xrightarrow{\star} Kbc$ where $K = \lambda x.\lambda y.x$.

8- Find M such that $M \xrightarrow{\star} Kab$ and $M \xrightarrow{\star} Kac$.

9- (difficult) Show that $\xleftarrow{\star}$ is not confluent.