Solutions of exercices

class 3-1

jean-jacques.levy@inria.fr

http://moscova.inria.fr/~levy/course/tsinghua/lambda

Exercice 1

Write following terms in long notation:

$\lambda x.x,\ \lambda x.\lambda y.x,\ \lambda xy.x,\ \lambda xyz.y,\ \lambda xyz.zxy, \lambda xyz.z(xy),$

$(\lambda x.\lambda y.x)MN,\ (\lambda xy.x)MN,\ (\lambda xy.y)MN,\ (\lambda xy.y)(MN)$

Solution

$\lambda x.x$	is for	$(\lambda x.x)$
$\lambda x.\lambda y.x$	is for	$(\lambda x.(\lambda y.x))$
$\lambda xy.x$	is for	$(\lambda x.(\lambda y.x))$
$\lambda xyz.y$	is for	$(\lambda x.(\lambda y.(\lambda z.y)))$
$\lambda xyz.zxy$	is for	$(\lambda x.(\lambda y.(\lambda z.((zx)y))))$
$(\lambda x.\lambda y.x)MN$	is for	$(((\lambda x.(\lambda y.x))M)N)$
$(\lambda xy.x)MN$	is for	$(((\lambda x.(\lambda y.x))M)N)$
$(\lambda xy.y)MN$	is for	$(((\lambda x.(\lambda y.y))M)N)$
$(\lambda xy.y)(MN)$	is for	$((\lambda x.(\lambda y.y))(MN))$

Exercice 3

Lemma $\mathsf{var}(M\{x := N\}) \subset (\mathsf{var}(M) - \{x\}) \cup \mathsf{var}(N)$

Proof

By structural induction on M.
Let write $lhs = \mathsf{var}(M\{x := N\})$ and $rhs = (\mathsf{var}(M) - \{x\}) \cup \mathsf{var}(N)$.
Case 1: $M = x$. Then $lhs = \mathsf{var}(N) = \emptyset \cup \mathsf{var}(N) = rhs$.

Case 2: $M = y$. Then $lhs = \{y\} \subset \{y\} \cup \mathsf{var}(N) = rhs$.

Case 3: $M = M_1M_2$. Then $lhs = \mathsf{var}(M_1\{x := N\}) \cup VAR(M_2\{x := N\})$ and $rhs = ((\mathsf{var}(M_1) \cup \mathsf{var}(M_2)) - \{x\}) \cup \mathsf{var}(N)$. Easy by induction.

Case 4: $M = \lambda y.M_1$. We may assume y enough fresh such that $M\{x := N\} = (\lambda y.M_1\{x := N\})$. Thus $lhs = \mathsf{var}(M_1\{x := N\}) - \{y\}$ and
$rhs = ((\mathsf{var}(M_1) - \{y\}) - \{x\}) \cup \mathsf{var}(N) = ((\mathsf{var}(M_1) - \{x\}) - \{y\}) \cup \mathsf{var}(N) = ((\mathsf{var}(M_1) - \{x\}) \cup \mathsf{var}(N)) - \{y\}$ since $y \notin \mathsf{var}(N)$.
Induction gives result.
QED.

Exercice 3 (cont'd)

Lemma $\mathsf{var}(M\{x := N\}) \subset (\mathsf{var}(M) - \{x\}) \cup \mathsf{var}(N)$

Proof

By structural induction on M.
Let write $lhs = \mathsf{var}(M\{x := N\})$ and $rhs = (\mathsf{var}(M) - \{x\}) \cup \mathsf{var}(N)$.
Case 1: $M = x$. Then $lhs = \mathsf{var}(N) = \emptyset \cup \mathsf{var}(N) = rhs$.

Case 2: $M = y$. Then $lhs = \{y\} \subset \{y\} \cup \mathsf{var}(N) = rhs$.

Case 3: $M = M_1M_2$. Then $lhs = \mathsf{var}(M_1\{x := N\}) \cup VAR(M_2\{x := N\})$ and $rhs = ((\mathsf{var}(M_1) \cup \mathsf{var}(M_2)) - \{x\}) \cup \mathsf{var}(N)$. Easy by induction.

Case 4: $M = \lambda y.M_1$. We may assume y enough fresh such that $M\{x := N\} = (\lambda y.M_1\{x := N\})$. Thus $lhs = \mathsf{var}(M_1\{x := N\}) - \{y\}$ and
$rhs = ((\mathsf{var}(M_1) - \{y\}) - \{x\}) \cup \mathsf{var}(N) = ((\mathsf{var}(M_1) - \{x\}) - \{y\}) \cup \mathsf{var}(N) = ((\mathsf{var}(M_1) - \{x\}) \cup \mathsf{var}(N)) - \{y\}$ since $y \notin \mathsf{var}(N)$.
Induction gives result.
QED.

Exercice 3 (cont'd)

- Find terms M such that:

$$M \longrightarrow M$$

$$M = M_0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow \cdots M_n = M \quad (M_i \text{ all distinct})$$

$$M =_\beta x\, M$$

$$M =_\beta \lambda x.M$$

$$M =_\beta MM$$

$$M =_\beta MN_1N_2\cdots N_n \text{ for all } N_1, N_2, \ldots N_n$$

Solutions

$M = \Delta\Delta$ with $\Delta = \lambda x.xx$
$M = Y_I = (\lambda x.I(xx))(\lambda x.I(xx))$ with $I = \lambda x.x$
$M = YM$ with $Y = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$
$M = YK$ with $K = \lambda xy.x$
$M = Y\Delta$
$M = YK$.

Exercice 3 (cont'd)

- Find term Y such that, for any M:

$$YM =_\beta M(YM)$$

Solution

$$Y = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$$

$$\begin{aligned} YM &= (\lambda f.(\lambda x.f(xx))(\lambda x.f(xx)))M \\ &\longrightarrow (\lambda x.M(xx))(\lambda x.M(xx)) \\ &\longrightarrow M((\lambda x.M(xx))(\lambda x.M(xx))) \\ &\longleftarrow M(YM) \end{aligned}$$

- Find Y' such that, for any M:

$$Y'M \xrightarrow{*} M(Y'M)$$

Solution

Try $Y' = AB$.
Then $Y'M = ABM$ and $A = \lambda bm.m(Abm)$.
If we take $A = B$, then
$Y' = AA$ with $A = \lambda bm.m(bbm)$ works.

$$\begin{aligned} Y'M &= (\lambda xy.y(xxy))(\lambda xy.y(xxy))M \\ &\xrightarrow{*} M((\lambda xy.y(xxy))(\lambda xy.y(xxy))M) \end{aligned}$$