Lambda-Calculus (II)

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Plan

- local confluence
- Church Rosser theorem
- Redexes and residuals
- Finite developments theorem
- Standardization theorem
Confluence
Consistency

**Question:** Can we get $M \xrightarrow{*} 2$ and $M \xrightarrow{*} 3$ ?

**Consequence:** $2 =_{\beta} 3$ !!
Question: If $M \equiv_{\beta} N$, then $M \rightarrow^* P$ and $N \rightarrow^* P$ for some $P$ ??

Then impossible to get $2 \equiv_{\beta} 3$
Confluency

**Question:** If $M \xrightarrow{*} N$ and $M \xrightarrow{*} P$, then $N \xrightarrow{*} Q$ and $P \xrightarrow{*} Q$ for some $Q$?

**Corollary:** [unicity of normal forms]

If $M \xrightarrow{*} N$ in normal form and $M \xrightarrow{*} N'$ in normal form, then $N = N'$. 
Confluency

Goal: If \( M \rightarrow^* N \) and \( M \rightarrow^* P \), there is \( Q \) such that \( N \rightarrow^* Q \) and \( P \rightarrow^* Q \)

How to prove confluency?
Local confluence

• **Theorem 1:** If $M \rightarrow N$ and $M \rightarrow P$ there is $Q$ such that $N \xrightarrow{*} Q$ and $P \xrightarrow{*} Q$

• **Lemma 1:** $M \rightarrow N$ implies $P\{x := M\} \xrightarrow{*} P\{x := N\}$

• **Lemma 2:** $M \rightarrow N$ implies $M\{x := P\} \rightarrow N\{x := P\}$

• **Substitution lemma:** $M\{x := N\}\{y := P\} = M\{y := P\}\{x := N\}\{y := P\}$

  when $x$ not free in $P$

• **Example:** $(\lambda x.xx)(Iz) \rightarrow (\lambda x.xx)z \rightarrow Iz(Iz) \xrightarrow{*} zz$

  where $I = \lambda x.x$
Confluency

- **Fact**: local confluency does not imply confluency
Confluency

We define \( \Rightarrow \) such that \( \rightarrow \subseteq \Rightarrow \subseteq \ast \rightarrow \)

- **Definition [parallel reduction]:**

  \[
  \begin{align*}
  \text{[Var Axiom]} & \quad x \Rightarrow x \\
  \text{[Const Axiom]} & \quad c \Rightarrow c \\
  \text{[App Rule]} & \quad \frac{M \Rightarrow M' \quad N \Rightarrow N'}{MN \Rightarrow M'N'} \\
  \text{[Abs Rule]} & \quad \frac{M \Rightarrow M'}{\lambda x.M \Rightarrow \lambda x.M'} \\
  \text{[//Beta Rule]} & \quad \frac{M \Rightarrow M' \quad M' \Rightarrow N' \quad N \Rightarrow N'}{(\lambda x.M)N \Rightarrow M'\{x := N'\}}
  \end{align*}
  \]

- **Example:**

  \[
  \begin{align*}
  x \Rightarrow x & \quad z \Rightarrow z \\
  \Rightarrow & \quad \Rightarrow \\
  Iz \Rightarrow z & \quad Iz \Rightarrow z \\
  Iz(Iz) \Rightarrow zz \\
  \end{align*}
  \]

  \( I = \lambda x.x \)
Confluency

• Goal is to prove **strongly local confluency**:

Example: \((\lambda x.xx)(iz)\) \(\not\Rightarrow\) \((\lambda x.xx)z\) \(\not\Rightarrow\) \(lz(lz)\) \(\not\Rightarrow\) \(zz\)
Confluency

- Proof of confluency:
Confluence

- **Lemma 4:** $M \not\rightarrow N$ and $P \not\rightarrow Q$ implies $M\{x := P\} \not\rightarrow N\{x := Q\}$

**Proof:** by structural induction on $M$.

Case 1: $M = x \not\rightarrow x = N$. Then $M\{x := P\} = P \not\rightarrow Q = N\{x := Q\}$

Case 2: $M = y \not\rightarrow y = N$. Then $M\{x := P\} = y \not\rightarrow y = N\{x := Q\}$

Case 3: $M = \lambda y.M_1 \not\rightarrow \lambda y.N_1 = N$ with $M_1 \not\rightarrow N_1$. By induction $M_1\{x := P\} \not\rightarrow N_1\{x := Q\}$. So $M\{x := P\} = \lambda y.M_1\{x := P\} \not\rightarrow \lambda y.N_1\{x := Q\} = N$.

Case 4: $M = M_1M_2 \not\rightarrow N_1N_2 = N$ with $M_1 \not\rightarrow N_1$ and $M_2 \not\rightarrow N_2$. By induction $M_1\{x := P\} \not\rightarrow N_1\{x := Q\}$ and $M_2\{x := P\} \not\rightarrow N_2\{x := Q\}$. So $M\{x := P\} = M_1\{x := P\}M_2\{x := P\} \not\rightarrow N_1\{x := Q\}N_2\{x := Q\} = N\{x := Q\}$.

Case 5: $M = (\lambda y.M_1)M_2 \not\rightarrow N_1\{y := N_2\} = N$ with $M_1 \not\rightarrow N_1$ and $M_2 \not\rightarrow N_2$. By induction $M_1\{x := P\} \not\rightarrow N_1\{x := Q\}$ and $M_2\{x := P\} \not\rightarrow N_2\{x := Q\}$. So $M\{x := P\} = (\lambda y.M_1\{x := P\})(M_2\{x := P\}) \not\rightarrow N_1\{x := Q\}\{y := N_2\{x := Q\}\} = N_1\{y := N_2\}\{x := Q\} = N$ by substitution lemma, since $y \notin \text{var}(Q) \subset \text{var}(P)$. 

Confluency

• **Lemma 5:** If $M \rightarrow N$ and $M \rightarrow P$, then $N \rightarrow Q$ and $N \rightarrow Q$ for some $Q$.

**Proof:** by structural induction on $M$.

Case 1: $M = x$. Then $M = x \rightarrow x = N$ and $M = x \rightarrow x = P$. We have too $N \rightarrow x = Q$ and $P \rightarrow x = Q$.

Case 2: $M = \lambda y.M_1 \rightarrow \lambda y.N_1 = N$ with $M_1 \rightarrow N_1$. Same for $M = \lambda y.M_1 \rightarrow \lambda y.P_1 = P$ with $M_1 \rightarrow P_1$. By induction $N_1 \rightarrow Q_1$ and $P_1 \rightarrow Q_1$ for some $Q_1$. So $N = \lambda y.N_1 \rightarrow \lambda y.Q_1 = Q$ and $P = \lambda y.P_1 \rightarrow \lambda y.Q_1 = Q$.

Case 3: $M = M_1 M_2 \rightarrow N_1 N_2 = N$ and $M = M_1 M_2 \rightarrow P_1 P_2 = P$ with $M_i \rightarrow N_i, M_i \rightarrow P_i (1 \leq i \leq 2)$. By induction $N_i \rightarrow Q_i$ and $P_i \rightarrow Q_i$ for some $Q_i$. So $N \rightarrow Q_1 Q_2 = Q$ and $P \rightarrow Q_1 Q_2 = Q$.

Case 4: $M = (\lambda x.M_1)M_2 \rightarrow N_1\{x := N_2\} = N$ and $M = (\lambda x.M_1)M_2 \rightarrow P' P_2 = P$ with $M_i \rightarrow N_i (1 \leq i \leq 2)$ and $\lambda x.M_1 \rightarrow P', M_2 \rightarrow P_2$. Therefore $P' = \lambda x.P_1$ with $M_1 \rightarrow P_1$. By induction $N_i \rightarrow Q_i$ and $P_i \rightarrow Q_i$ for some $Q_i$. So $N \rightarrow Q_1\{x := Q_2\} = Q$ by lemma 4. And $P \rightarrow Q_1\{x := Q_2\} = Q$ by definition.

Case 5: symmetric.
Confluence

Proof: ....

Case 6: $M = (\lambda x.M_1)M_2 \rightarrow N_1\{x := N_2\} = N$ and $M = (\lambda x.M_1)M_2 \rightarrow P_1\{x := P_2\} = P$ with $M_i \rightarrow N_i, M_i \rightarrow P_i$ (1 ≤ $i$ ≤ 2). By induction $N_i \rightarrow Q_i$ and $P_i \not\rightarrow Q_i$ for some $Q_i$.

So $N \rightarrow Q_1\{x := Q_2\} = Q$ and $P \rightarrow Q_1\{x := Q_2\} = Q$ by lemma 4. □

• **Lemma 6:** If $M \rightarrow N$, then $M \not\rightarrow N$.

• **Lemma 7:** If $M \not\rightarrow N$, then $M \rightarrow^* N$.

**Proofs:** obvious.

• **Theorem 2 [Church-Rosser]:**

If $M \rightarrow^* N$ and $M \rightarrow^* P$, then $N \rightarrow^* Q$ and $P \rightarrow^* Q$ for some $Q$. 
Confluency

• previous axiomatic method is due to Martin-Löf
• Martin-Löf’s method models inside-out parallel reductions
• there are other proofs with explicit redexes

• Curry’s finite developments
Residuals of redexes

- tracking redexes while contracting others
- examples:

\[ \Delta(la) \rightarrow la(la) \]
\[ la(\Delta(lb)) \rightarrow la(lb(lb)) \]
\[ l(\Delta(la)) \rightarrow l(la(la)) \]
\[ \Delta(la) \rightarrow la(la) \]
\[ la(\Delta(lb)) \rightarrow la(lb(lb)) \]
\[ \Delta\Delta \rightarrow \Delta\Delta \]
\[ (\lambda x. la)(lb) \rightarrow la \]

\[ \Delta = \lambda x. xx \quad l = \lambda x. x \quad K = \lambda xy.x \]
Residuals of redexes

- when $R$ is redex in $M$ and $M \xrightarrow{S} N$
  
  the set $R/S$ of residuals of $R$ in $N$ is defined by inspecting relative positions of $R$ and $S$ in $M$:

1- $R$ and $S$ disjoint, $M = \cdots R \cdots S \cdots \xrightarrow{S} \cdots R \cdots S' \cdots = N$

2- $S$ in $R = (\lambda x.A)B$
  
  2a- $S$ in $A$, $M = \cdots (\lambda x. \cdots S \cdots)B \cdots \xrightarrow{S} \cdots (\lambda x. \cdots S' \cdots)B \cdots = N$
  
  2b- $S$ in $B$, $M = \cdots (\lambda x.A)(\cdots S \cdots) \cdots \xrightarrow{S} \cdots (\lambda x.A)(\cdots S' \cdots) \cdots = N$

3- $R$ in $S = (\lambda y.C)D$
  
  3a- $R$ in $C$, $M = \cdots (\lambda y. \cdots R \cdots)D \cdots \xrightarrow{S} \cdots R\{y := D\} \cdots = N$
  
  3b- $R$ in $D$, $M = \cdots (\lambda y.C)(\cdots R \cdots) \cdots \xrightarrow{S} \cdots (\cdots R \cdots)(\cdots R \cdots) \cdots = N$

4- $R$ is $S$, no residuals of $R$. 
Residuals of redexes

• when $\rho$ is a reduction from $M$ to $N$, i.e. $\rho : M \rightarrow* N$
  the set of residuals of $R$ by $\rho$ is defined by **transitivity** on the length of $\rho$
  and is written $R/\rho$

• notice that we can have $S \in R/\rho$ and $R \neq S$
  residuals may **not** be syntactically **equal** (see previous 3rd example)

• residuals **depend on reductions**. Two reductions between same terms may
  produce two distinct sets of residuals.

• a redex is residual of a **single** redex (the inverse of the residual relation is a
  function): $R \in S/\rho$ and $R \in T/\rho$ implies $S = T$
Exercises

• Find redex \( R \) and reductions \( \rho \) and \( \sigma \) between \( M \) and \( N \) such that residuals of \( R \) by \( \rho \) and \( \sigma \) differ. Hint: consider \( M = I(lx) \)

• Show that residuals of nested redexes keep nested.

• Show that residuals of disjoint redexes may be nested.

• Show that residuals of a redex may be nested after several reduction steps.

Created redexes

• A redex is created by reduction \( \rho \) if it is not a residual by \( \rho \) of a redex in initial term. Thus \( R \) is created by \( \rho \) when \( \rho : M \rightarrow* N \) and \( \not\exists S, R \in S/\rho \)

\[
(\lambda x.xa)l \rightarrow l a \\
(\lambda xy.xy)ab \rightarrow (\lambda y.ay)b
\]

\[
lla \rightarrow la \\
\Delta\Delta \rightarrow \Delta\Delta
\]
Residuals of redexes

\[(\lambda x.xx)((\lambda f.3)(\lambda x.x))\]

\[(\lambda f.3)(\lambda x.x)((\lambda f.3)(\lambda x.x))\]

\[(\lambda f.3)(\lambda x.x)((\lambda x.x)3)\]

\[(\lambda x.x)((\lambda x.x)3)\]

\[(\lambda x.x)3((\lambda f.3)(\lambda x.x))\]

\[(\lambda f.3)(\lambda x.x)3\]

\[(\lambda x.x)3((\lambda x.x)3)\]

\[3((\lambda f.3)(\lambda x.x))\]

\[(\lambda x.x)33\]

\[3((\lambda x.x)3)\]

\[(\lambda x.xx)3\]

\[33\]
Relative reductions

\[(\lambda x.x)((\lambda f.3)(\lambda x.x))\]

\[(\lambda f.3)(\lambda x.x)((\lambda f.3)(\lambda x.x))\]

\[(\lambda f.3)(\lambda x.x)(\lambda x.x)3\]

\[(\lambda x.x)3((\lambda f.3)(\lambda x.x))\]

\[(\lambda x.x)((\lambda x.x)3)\]

\[(\lambda x.x)3((\lambda x.x)3)\]

\[3((\lambda f.3)(\lambda x.x))\]

\[(\lambda x.x)33\]

\[3((\lambda x.x)3)\]

\[(\lambda x.x)3\]

\[33\]
Finite developments

• Let $\mathcal{F}$ be a set of redexes in $M$. A reduction relative to $\mathcal{F}$ only contracts residuals of $\mathcal{F}$.

• When there are no more residuals of $\mathcal{F}$ to contract, we say the relative reduction is a development of $\mathcal{F}$.

• **Theorem 3 [finite developments] (Curry)** Let $\mathcal{F}$ be a set of redexes in $M$. Then:
  - relative reductions cannot be infinite; they all end in a development of $\mathcal{F}$
  - all developments end on a same term $N$
  - let $R$ be a redex in $M$. Then residuals of $R$ by finite developments of $\mathcal{F}$ are the same.
Finite developments

• Therefore we can define (without ambiguity) a new parallel step reduction:

\[ \rho : M \overset{\mathcal{F}}{\rightarrow} N \]

and when \( R \) is a redex in \( M \), we can write \( R/\mathcal{F} \) for its residuals in \( N \)

• Two corollaries:

Lemma of Parallel Moves

Cube Lemma
Labeled calculus

• Finite developments will be shown with a labeled calculus.

• Lambda calculus with labeled redexes

\[ M, N, P ::= x, y, z, \ldots \quad \text{(variables)} \]
\[ | (\lambda x. M) \quad \text{($M$ as function of $x$)} \]
\[ | (M \ N) \quad \text{($M$ applied to $N$)} \]
\[ | c, d, \ldots \quad \text{(constants)} \]
\[ | (\lambda x. M)^r \ N \quad \text{(labeled redexes)} \]

• $\mathcal{F}$-labeled reduction

\[ (\lambda x. M)^r N \rightarrow M\{x := N\} \quad \text{when $r \in \mathcal{F}$} \]

• Labeled substitution

\[ \ldots \text{as before} \]
\[ ((\lambda x. M)^r N)\{y := P\} = ((\lambda x. M)\{y := P\})^r (N\{y := P\}) \]
Labeled calculus

• **Theorem** For any $\mathcal{F}$, the labeled calculus is **confluent**.

• **Theorem** For any $\mathcal{F}$, the labeled calculus is **strongly normalizable** (no infinite labeled reductions).

• **Lemma** For any $\mathcal{F}$-reduction $\rho : M \rightarrow N$, a labeled redex in $N$ has label $r$ if and only if it is **residual** by $\rho$ of a redex with label $r$ in $M$.

• **Theorem 3 [finite developments] (Curry)**
Labeled calculus

• Proof of confluency is again with Martin-Löf’s axiomatic method.
• Proof of residual property is by simple inspection of a reduction step.
• Proof of termination is slightly more complex with following lemmas:

• **Notation** $M \xrightarrow{\text{int}} N$ if $M$ reduces to $N$ without contracting a toplevel redex.

• **Lemma 1** [Barendregt-like] $M\{x := N\} \xrightarrow{\text{int}} (\lambda y. P)^r Q$ implies

  $M = (\lambda y. A)^r B$ with $A\{x := N\} \rightarrow^* P$, $B\{x := N\} \rightarrow^* Q$

  or

  $M = x$ and $N \rightarrow^* (\lambda y. P)^r Q$

• **Lemma 2** $M, N \in S\mathcal{N}$ (strongly normalizing) implies $M\{x := N\} \in S\mathcal{N}$

• **Theorem** $M \in S\mathcal{N}$ for all $M$. 

Labeled calculus proofs

- **Lemma 1** [Barendregt-like] \( M\{x := N\} \rightarrow^* (\lambda y. P)^r Q \) implies

\[
M = (\lambda y. A)^r B \text{ with } A\{x := N\} \rightarrow^* P, B\{x := N\} \rightarrow^* Q
\]

or

\[
M = x \text{ and } N \rightarrow^* (\lambda y. P)^r Q
\]

**Proof** Let \( P^* \) be \( P\{x := N\} \) for any \( P \).

Case 1: \( M = x \). Then \( M^* = N \) and \( N \rightarrow^* (\lambda y. P)^r Q \).

Case 2: \( M = y \). Then \( M^* = y \). Impossible.

Case 2: \( M = \lambda y. M_1 \). Again impossible.

Case 3: \( M = M_1 M_2 \) or \( M = (\lambda y. M_1)^s M_2 \) with \( s \neq r \). These cases are also impossible.

Case 4: \( M = (\lambda y. M_1)^r M_2 \). Then \( M_1^* \rightarrow^* P \) and \( M_2^* \rightarrow^* Q \).
Labeled calculus proofs

- **Lemma 2** $M, N \in SN$ (strongly normalizing) implies $M\{x := N\} \in SN$

**Proof:** by induction on $\langle\text{depth}(M), ||M||\rangle$. Let $P^*$ be $P\{x := N\}$ for any $P$.

Case 1: $M = x$. Then $M^* = N \in SN$. If $M = y$. Then $M^* = y \in SN$.

Case 2: $M = \lambda y.M_1$. Then $M^* = \lambda y.M_1^*$ and by induction $M_1^* \in SN$.

Case 3: $M = M_1 M_2$ and never $M^* \xrightarrow{\text{int}} (\lambda y.A)^r B$. Same argument on $M_1$ and $M_2$.

Case 4: $M = M_1 M_2$ and $M^* \xrightarrow{\text{int}} (\lambda y.A)^r B$. We can always consider first time when this toplevel redex appears. Hence we have $M^* \xrightarrow{\text{int}} (\lambda y.A)^r B$. By lemma 1, we have two cases:

Case 4.1: $M = (\lambda y.M_3)^r M_2$ with $M_3^* \xrightarrow{\text{int}} A$ and $M_2^* \xrightarrow{\text{int}} B$. Then $M^* = (\lambda y.M_3^*)^r M_2^*$. As $M_3 \in SN$ and $M_2 \in SN$, the internal reductions from $M^*$ terminate by induction. If $r \not\in F$, there are no extra reductions. If $r \in F$, we can have $M_3^* \xrightarrow{\text{int}} A$, $M_2^* \xrightarrow{\text{int}} B$ and $(\lambda y.A)^r B \xrightarrow{\text{int}} A\{y := B\}$. But $M \xrightarrow{\text{int}} M_3\{y := M_2\}$ and $(M_3\{y := M_2\})^* \xrightarrow{\text{int}} A\{y := B\}$. As depth($A\{y := B\} < \text{depth}(A\{y := B\}) < \text{depth}(M)$, we get $A\{y := B\} \in SN$ by induction.

Case 4.2: $M = x$. Impossible.
Labeled calculus proofs

• **Theorem**  \( M \in SN \) for all \( M \).

**Proof:** by induction on \( ||M|| \).

Case 1: \( M = x \). Obvious.

Case 2: \( M = \lambda x. M_1 \). Obvious since \( M_1 \in SN \) by induction.

Case 3: \( M = M_1 M_2 \) and \( M_1 \neq (\lambda x. A)^r \). Then all reductions are internal to \( M_1 \) and \( M_2 \). Therefore \( M \in SN \) by induction on \( M_1 \) and \( M_2 \).

Case 4: \( M = (\lambda x. M_1)^r M_2 \) and \( r \notin F \). Same argument on \( M_1 \) and \( M_2 \).

Case 5: \( M = (\lambda x. M_1)^r M_2 \) and \( r \in F \). Then \( M_1 \) and \( M_2 \) in \( SN \) by induction. But we can also have \( M \rightarrow^+ (\lambda x. A)^r B \rightarrow A\{x := B\} \) with \( A \) and \( B \) in \( SN \). By Lemma 2, we know that \( A\{x := B\} \in SN \).
Standardization
Standard reduction

Redex $R$ is **to the left of** redex $S$ if the $\lambda$ of $R$ is to the left of the $\lambda$ of $S$.

\[ M = \cdots (\lambda x. A)B \cdots (\lambda y. C)D \cdots \]

or

\[ M = \cdots (\lambda x. \cdots (\lambda y. C)D \cdots )B \cdots \]

or

\[ M = \cdots (\lambda x. A)(\cdots (\lambda y. C)D \cdots )\cdots \]

The reduction $M = M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} M_2 \cdots \xrightarrow{R_n} M_n = N$ is **standard** iff for all $i, j$ ($0 < i < j \leq n$), redex $R_j$ is not a residual of redex $R'_j$ to the left of $R_i$ in $M_{i-1}$.
Standard reduction

\[ M = (\lambda x.x)((\lambda f.f3)(\lambda x.x)) \]

\[ (\lambda f.f3)(\lambda x.x)(\lambda f.f3)(\lambda x.x) \]

\[ (\lambda f.f3)(\lambda x.x)((\lambda x.x)3) \]

\[ (\lambda x.x)((\lambda x.x)3) \]

\[ (\lambda x.x)3((\lambda x.x)3) \]

\[ 3((\lambda f.f3)(\lambda x.x)) \]

\[ (\lambda x.x)(\lambda x.x)3 \]

\[ N = 3((\lambda x.x)3) \]

\[ (\lambda x.x)33 \]

\[ (\lambda x.x)33 \]

\[ 33 \]
Standardization

- **Theorem [standardization] (Curry)**  Any reduction can be standardized.

- The **normal reduction** (each step contracts the leftmost-outermost redex) is a standard reduction.

- **Corollary [normalization]**  If $M$ has a normal form, the normal reduction reaches the normal form.
Standardization lemma

- **Notation:** write $R <_\ell S$ if redex $R$ is to the left of redex $S$.

- **Lemma 1** Let $R, S$ be redexes in $M$ such that $R <_\ell S$. Let $M \xrightarrow{S} N$. Then $R/S = \{R'\}$. Furthermore, if $T' <_\ell R'$, then $\exists T, T <_\ell R, T' \in T/S$. [one cannot create a redex through another more-to-the-left]

\[
\begin{align*}
M &\xrightarrow{S} N \\
R &\quad R' \\
\end{align*}
\]

- **Proof of standardization thm:** [Klop] application of the finite developments theorem and previous lemma.
Standardization axioms

• 3 axioms are sufficient to get lemma 1

• **Axiom 1 [linearity]**  $S <_{\ell} R$  implies  $\exists! R', R' \in R/S$

• **Axiom 2 [context-freeness]**  $S <_{\ell} R$ and $R' \in R/S$ and $T' \in T/S$  implies  $T \not\preceq R$  iff  $T' \not\preceq R'$  where  $\not\preceq$ is $<_{\ell}$ or $>_{\ell}$

• **Axiom 3 [left barrier creation]**  $R <_{\ell} S$ and $\not\preceq T'$, $T \in T'/S$  implies  $R <_{\ell} T$
Standardization proof

• Proof:

Each square is an application of the lemma of parallel moves. Let $\rho_i$ be the horizontal reductions and $\sigma_j$ the vertical ones. Each horizontal step is a parallel step, vertical steps are either elementary or empty.

We start with reduction $\rho_0$ from $M$ to $N$. Let $R_1$ be the leftmost redex in $M$ with residual contracted in $\rho_0$. By lemma 1, it has a single residual $R'_1$ in $M_1$, $M_2$, ... until it belongs to some $F_k$. Here $R'_1 \in F_2$. There are no more residuals of $R_1$ in $M_{k+1}$, $M_{k+2}$, ... .

Let $R_2$ be leftmost redex in $P_1$ with residual contracted in $\rho_1$. Here the unique residual is contracted at step $n$. Again with $R_3$ leftmost with residual contracted in $\rho_2$. Etc.
Standardization proof

• Proof (cont’d):

Now reduction $\sigma_0$ starting from $M$ cannot be infinite and stops for some $p$. If not, there is a rightmost column $\sigma_k$ with infinitely non-empty steps. After a while, this reduction is a reduction relative to a set $F_i^j$, which cannot be infinite by the Finite Development theorem.

Then $\rho_p$ is an empty reduction and therefore the final term of $\sigma_0$ is $N$. 
We claim $\sigma_0$ is a standard reduction. Suppose $R_k$ ($k > i$) is residual of $S_i$ to the left of $R_i$ in $P_{i-1}$.

By construction $R_k$ has residual $S^j_k$ along $\rho_{i-1}$ contracted at some $j$ step. So $S^j_k$ is residual of $S_i$.

By the cube lemma, it is also residual of some $S^j_i$ along $\sigma_{j-1}$. Therefore there is $S^j_i$ in $F^j_i$ residual of $S_i$ leftmore or outer than $R_i$.

Contradiction.
Redex creation
Created redexes

• A redex is **created by reduction** $\rho$ if it is not a residual by $\rho$ of a redex in initial term. Thus $R$ is created by $\rho$ when $\rho : M \rightarrow N$ and $\not\exists S$, $R \in S/\rho$

  $$(\lambda x. xa)I \rightarrow la$$
  $$(\lambda xy.xy)ab \rightarrow (\lambda y. ay)b$$
  $$lla \rightarrow la$$
  $$\Delta\Delta \rightarrow \Delta\Delta$$

• By Finite Developments thm, a reduction can be infinite iff it does not stop creating new redexes.

  $$\Delta\Delta \rightarrow \Delta\Delta \rightarrow \Delta\Delta \rightarrow \Delta\Delta \rightarrow \cdots$$

• If the length of creation is bounded, there is also a generalized finite developments theorem.
Created redexes in typed calculus

• only 2 cases for creation of redexes within a reduction step

\[(\lambda x. \cdots xN \cdots)(\lambda y. M) \rightarrow \cdots (\lambda y. M)N' \cdots\]

\(\sigma \rightarrow \tau\)

creates

\[(\lambda x.\lambda y. M)NP \rightarrow (\lambda y. M')P\]

\(\tau\)

\(\sigma \rightarrow \tau\)

creates

• length of creation is bounded by size of types of initial term
Other properties
Other properties

- confluence with **eta**-rules, **delta**-rules
- **generalized** finite developments theorem
- **permutation** equivalence
- redex **families**
- finite developments vs strong normalization
- completeness of reduction **strategies**
- **head** normal forms
- **Bohm** trees
- continuity theorem
- sequentiality of Bohm trees
- models of the type-free lambda-calculus
- **typed** lambda-calculi
- continuations and reduction strategies
- ...

- process calculi and lambda-calculus
- abstract reduction systems
- **explicit** substitutions
- implementation of functional languages
- lazy evaluators
- SOS
- all theory of **programming languages**
- ...

- connection to mathematical **logic**
- calculus of constructions
- ...

...
Homeworks
Exercises

• Show that:

1- \( M \xrightarrow{\eta} N \xrightarrow{\eta} P \) implies \( M \xrightarrow{} Q \xrightarrow{\eta} P \) for some \( Q \)

2- \( M \xrightarrow{*} N \xrightarrow{*} P \) implies \( M \xrightarrow{*} Q \xrightarrow{*} P \) for some \( Q \)

3- \( M \xrightarrow{\beta,\eta} N \) implies \( M \xrightarrow{*} P \xrightarrow{*} \eta N \) for some \( P \)

4- \( M \xrightarrow{} N \) and \( M \xrightarrow{\eta} P \) implies \( N \xrightarrow{*} Q \) and \( P \xrightarrow{1} Q \) for some \( Q \)

5- \( M \xrightarrow{*} N \) and \( M \xrightarrow{\eta} P \) implies \( N \xrightarrow{*} Q \) and \( P \xrightarrow{*} Q \) for some \( Q \)

6- \( M \xrightarrow{\beta,\eta} N \) and \( M \xrightarrow{\beta,\eta} P \) implies \( N \xrightarrow{*} Q \) and \( P \xrightarrow{*} Q \) for some \( Q \)

Therefore \( \xrightarrow{*} \) \( \beta,\eta \) is confluent.

• Show same property for \( \beta \)-reduction and \( \eta \)-expansion \( (\xrightarrow{} \cup \leftarrow \eta)^* \)
Exercices

7- Show there is no $M$ such that $M \xrightarrow{*} Kac$ and $M \xrightarrow{*} Kbc$ where $K = \lambda x.\lambda y.x$.

8- Find $M$ such that $M \xrightarrow{*} Kab$ and $M \xrightarrow{*} Kac$.

9- (difficult) Show that $\xrightarrow{*}$ is not confluent.

10- Show that $\Delta \Delta (\Pi)$ has no normal form when $I = \lambda x.x$ and $\Delta = \lambda x.xx$.

11- Show that $\Delta \Delta M_1 M_2 \cdots M_n$ has no normal form for any $M_1, M_2, \ldots M_n \ (n \geq 0)$.

12- Show there is no $M$ whose reduction graph is exactly following:

```
    M
   / \  \
  /   \  /
 M1   M2 M3
  |   |   |
  \   \   \       
    N
```

13- Show that rightmost-outermost reduction may miss normal forms.