Reductions and Causality (VI)

jean-jacques.levy@inria.fr
Escuela de Ciencias Informáticas
Universidad de Buenos Aires

July 26, 2013

http://jeanjacqueslevy.net/courses/13eci

Plan

- complete reductions
- sublattice of complete reductions
- more on canonical representatives
- costs of reductions + sharing
- speculative computations
- semantics with Böhm trees

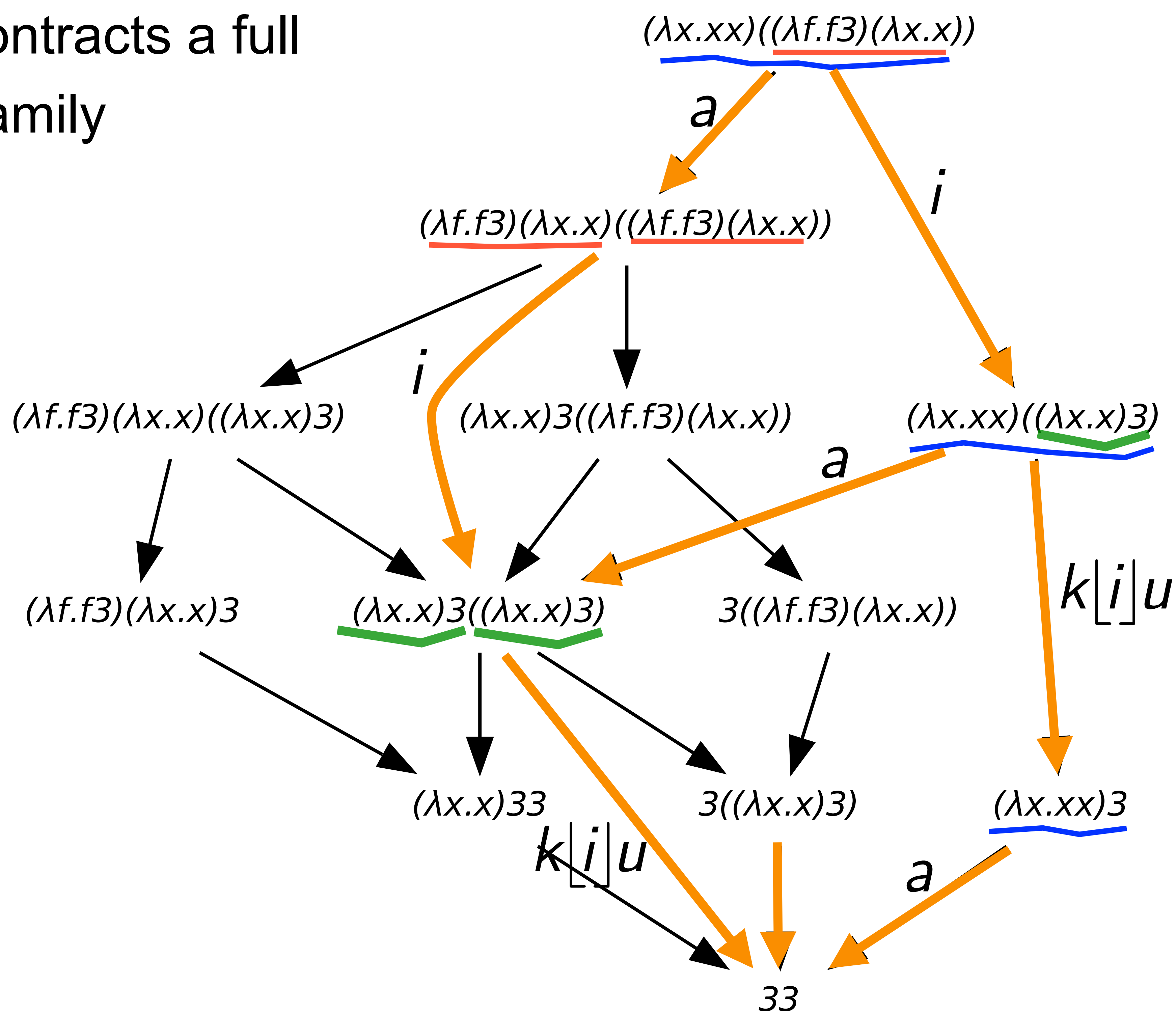
Labeled λ-calculus

CENTRE DE RECHERCHE COMMUN

INRIA MICROSOFT RESEARCH

Complete reductions (1/5)

- Each step contracts a full single redex family



Complete reductions (2/5)

- **Definition** `[complete reductions]`

  $\langle \rho, \mathcal{F} \rangle$ is an historical set of redexes when $\mathcal{F}$ is a set of redexes in final term of ρ.

  $\langle \rho, \mathcal{F} \rangle$ is **f-complete** when it is maximum set such that

  $$R, S \in \mathcal{F} \quad \text{implies} \quad \langle \rho, R \rangle \sim \langle \rho, S \rangle$$

  An **f-complete reduction** contracts an f-complete set at each step.

- **Proposition** `[lattice of f-complete reductions]`

  Complete reductions form a sub-lattice of the lattice of reductions.

  Proof simple use of following lemma which implies f-complete parallel moves.

Complete reductions (3/5)

- **Notations**

  $M \stackrel{\alpha}{\Longrightarrow} N$ when $M \stackrel{\mathcal{F}}{\longrightarrow} N$ and $\mathcal{F}$ is the set of redexes with name α in M.

  MaxRedNames(*M*) when all redexes in *M* have maximal names.

- **Lemma** `[complete reductions preserve max redex names]`

  $M \stackrel{\alpha}{\Longrightarrow} N$ and MaxRedNames(M) implies MaxRedNames(N)

Complete reductions (4/5)

- **Definition** [d-complete reductions]

  $\langle \rho, \mathcal{F} \rangle$ is **d-complete** when it is maximum set such that

  $$\langle \rho_0, R_0 \rangle \leq \langle \rho, \mathcal{F} \rangle \text{ for some } \langle \rho_0, R_0 \rangle$$

  An **d-complete reduction** contracts a d-complete set at each step.

- **Proposition** [below canonical representative]

  Let $\langle \rho_0, R_0 \rangle$ be canonical representative in its family.
  Let $\rho_0 \sqsubseteq \rho$. Then $\langle \rho_0, R_0 \rangle \sim \langle \rho, R \rangle$ iff $\langle \rho_0, R_0 \rangle \leq \langle \rho, R \rangle$.

  Proof difficult.

- **Proposition** [f-complete = d-complete]

  d-complete reductions coincide with f-complete reductions.

Complete reductions (5/5)

- **Proposition** `[length of reduction = number of families]`

  In complete reductions, number of steps equals the number of contracted redex families.

  Proof application of MaxRedNames lemma.

- **Corollary** `[optimal reductions]`

  In complete reductions, never redex of same family is contracted twice.

- **Implementation** `[optimal reductions]`

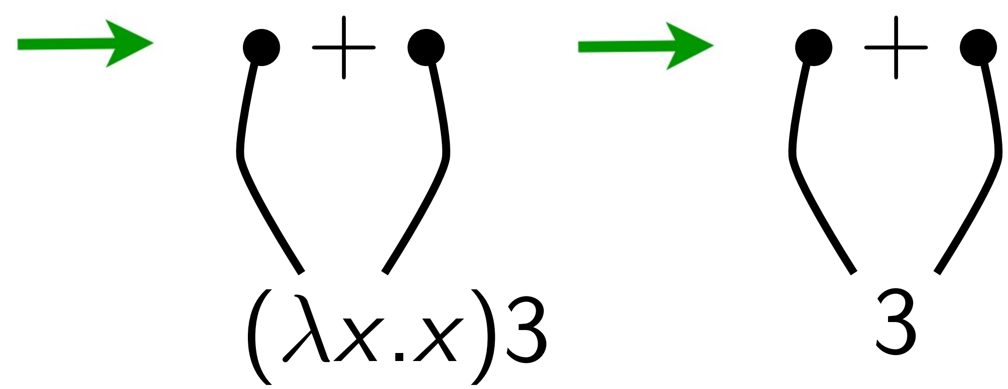
  Can we implement efficiently complete reductions ?

Implementation (1/5)

- **Implementation** [optimal reductions] algorithm [John Lamping, 90 -- Gonthier-Abadi-JJ, 91]

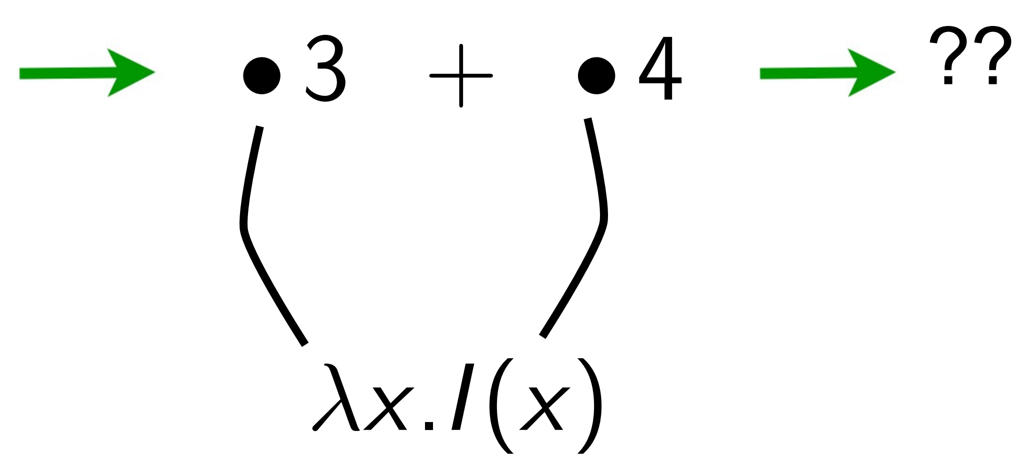
- Sharing of basic values is easy:

$$(\lambda x.x + x)((\lambda x.x)3)$$



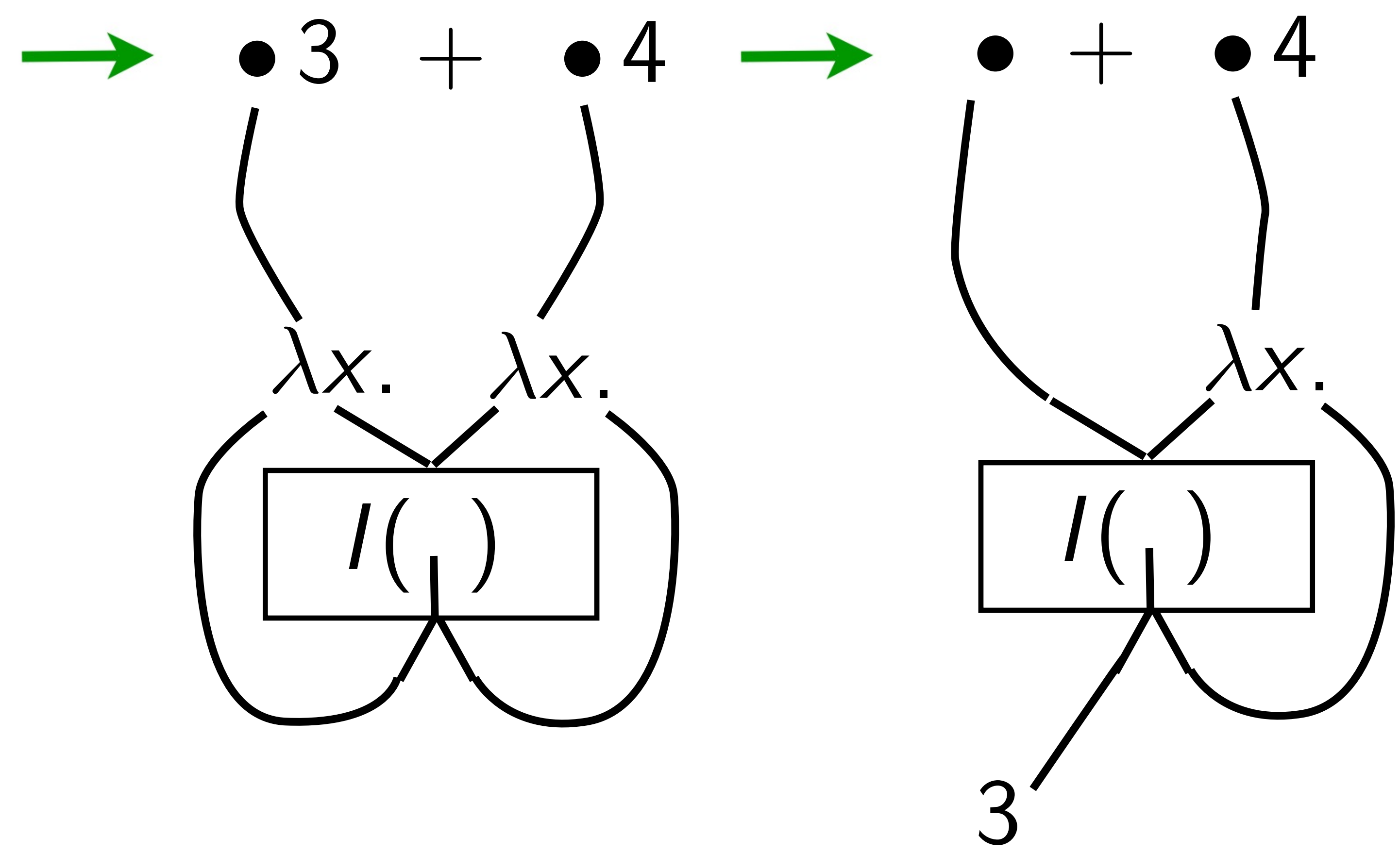
- Problem is sharing of functions:

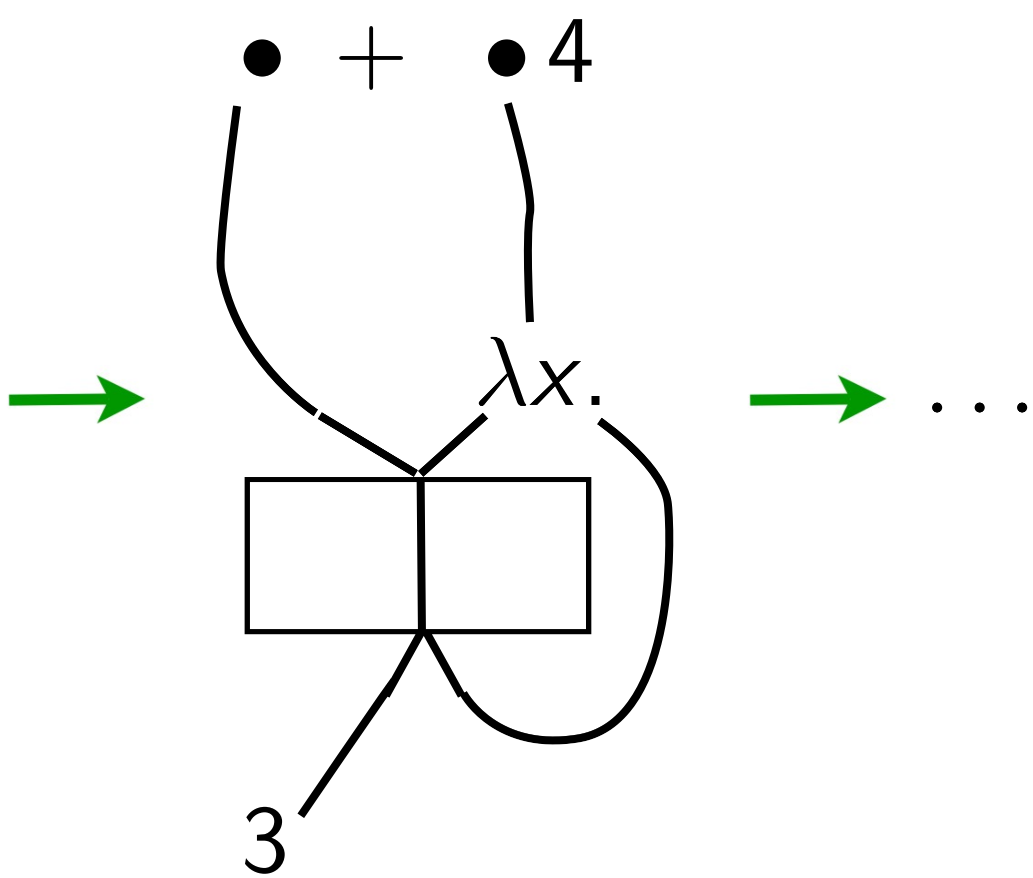
$$(\lambda x.x3 + x4)((\lambda x.I(x))$$



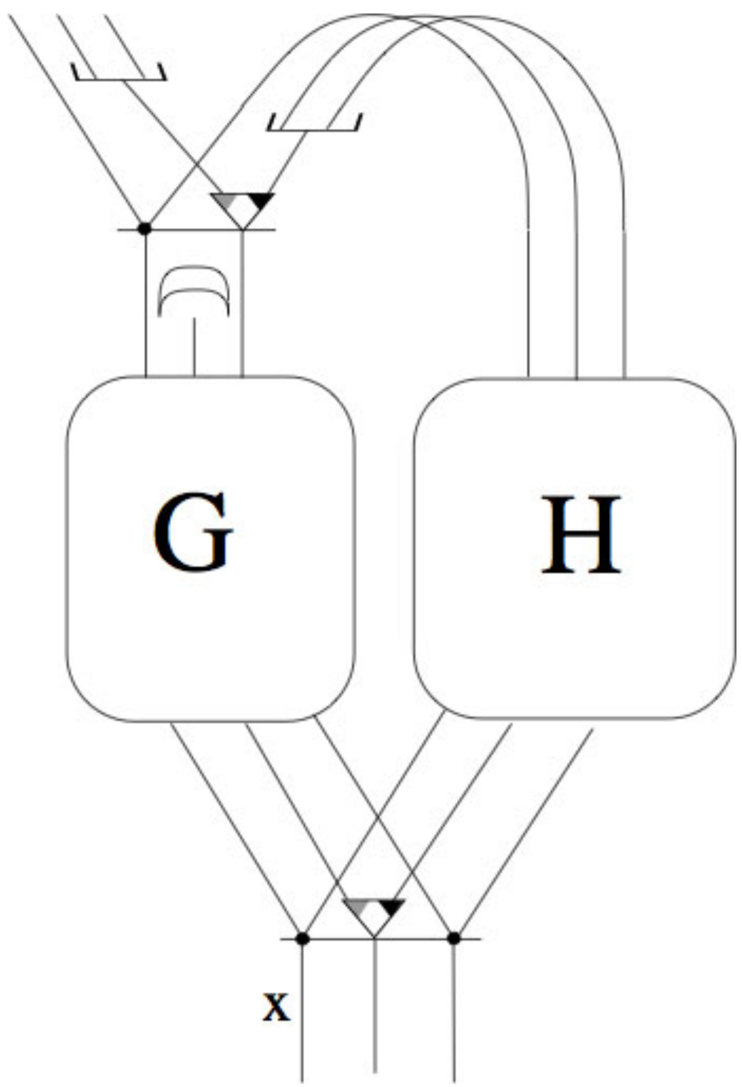
Implementation (2/5)

$(\lambda x.x3 + x4)((\lambda x.I(x))$

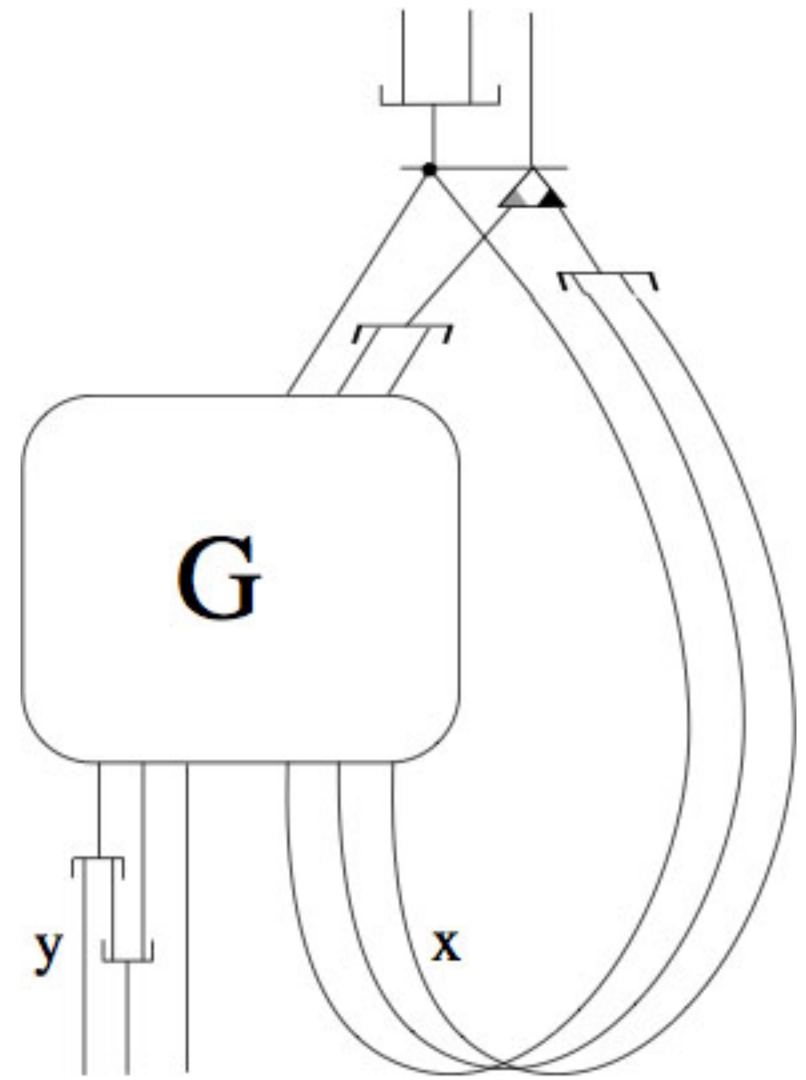




Implementation (3/5)

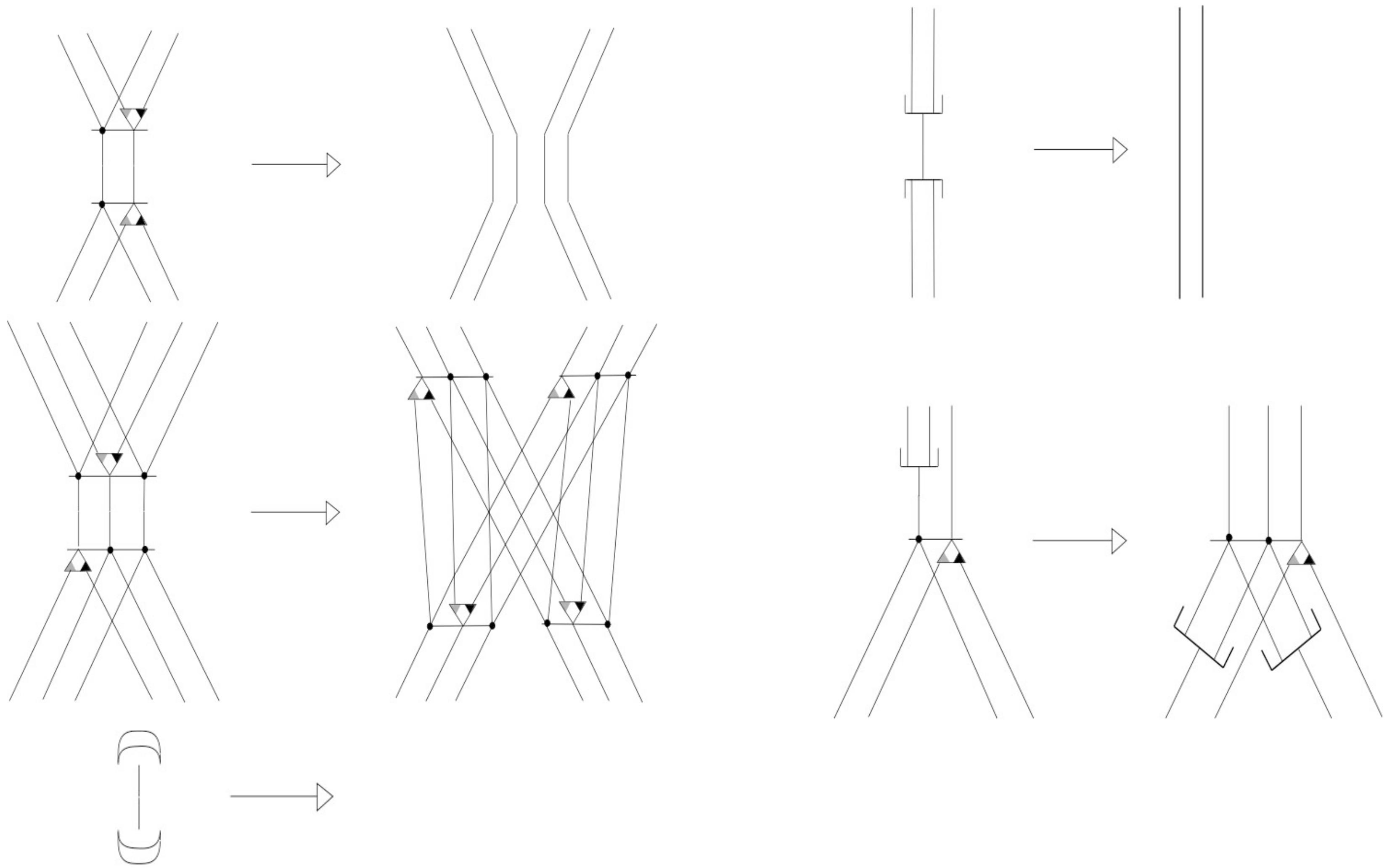


application

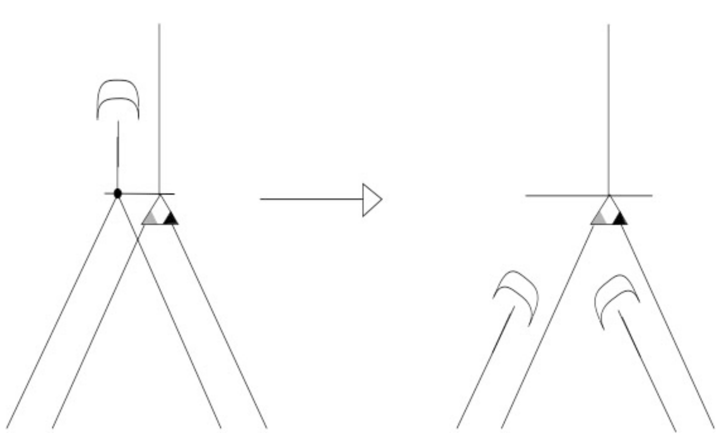


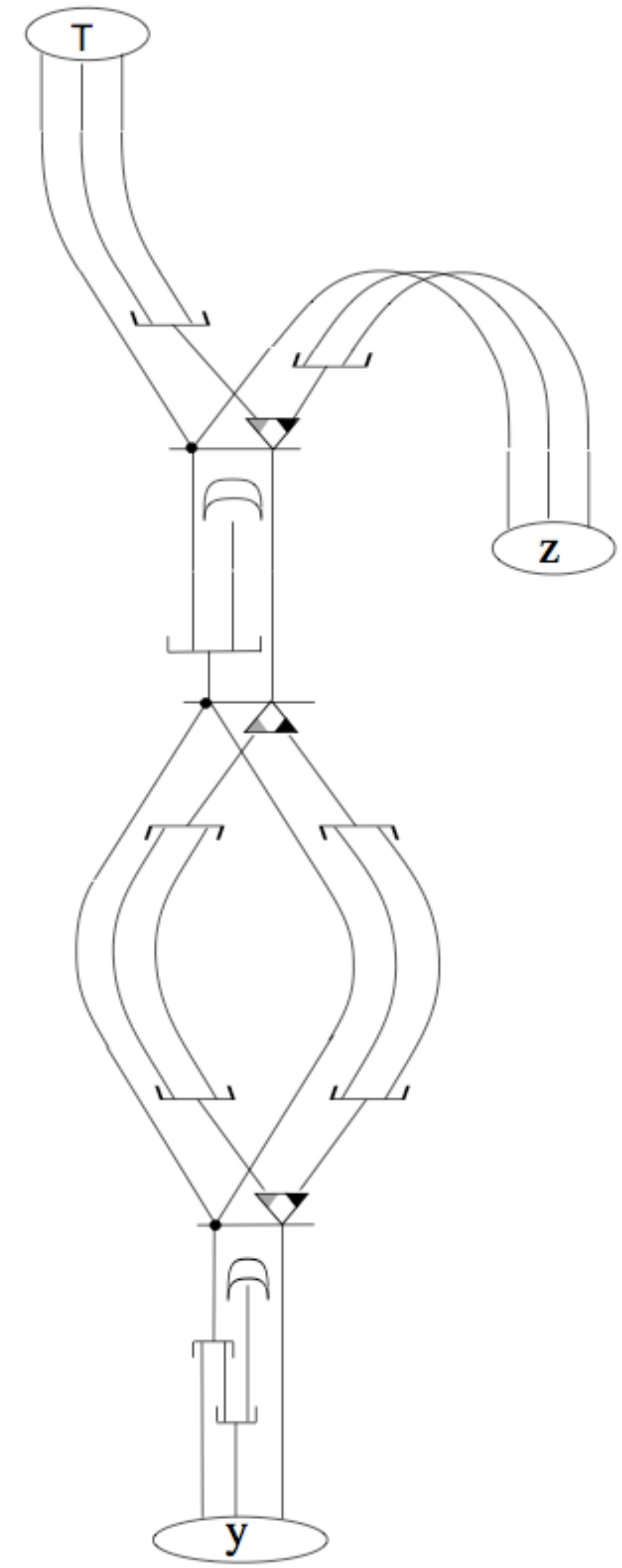
λ-abstraction

Implementation (4/5)



rules





$(\lambda x.yx)z$

Implementation (5/5)

- beautiful Lamping's algorithm is unpractical
- highly exponential in the handling of fans node (not elementary recursive) [Asperti, Mairson 2000]
- nice algorithms unsharing paths to bound variables [Wadsworth 92, Shivers-Wand 2010]
- Haskell, Coq, Caml ??

Speculative computations

CENTRE DE RECHERCHE COMMUN
INRIA MICROSOFT RESEARCH

Permutations in call by value

- **Definition** [call by value]

  A **value** remains a value if computed or substituted by a value

$$V ::= x \mid \lambda x.M$$

  The call-by-value reduction strategy is defined by:

$$(\lambda x.M)V \xrightarrow[\text{cbv}]{} M\{x := V\}$$

$$\frac{M \xrightarrow[\text{cbv}]{} M'}{MN \xrightarrow[\text{cbv}]{} M'N} \qquad \frac{N \xrightarrow[\text{cbv}]{} N'}{MN \xrightarrow[\text{cbv}]{} MN'}$$

- **Fact** [permutations in call by value]

  Equivalence by permutations only permute disjoint redexes.

Speculative reductions

- **Definition** [speculative call, Boudol-Petri 2010]

$$V \ ::= \ x \ | \ \lambda x.M$$

The speculative reduction strategy is defined by:

$$(\lambda x.M)V \xrightarrow[\text{spec}]{} M\{x := V\}$$

$$(\lambda x.M)N \xrightarrow[\text{spec}]{} (\lambda V?\, M\{x := V\})N$$

$$(\lambda V?\, M)V \xrightarrow[\text{spec}]{} M$$

$$\frac{M \xrightarrow[\text{spec}]{} M'}{MN \xrightarrow[\text{spec}]{} M'N} \qquad \frac{N \xrightarrow[\text{spec}]{} N'}{MN \xrightarrow[\text{spec}]{} MN'} \qquad \frac{M \xrightarrow[\text{spec}]{} M'}{\lambda V?M \xrightarrow[\text{spec}]{} \lambda V?M'}$$

Assigning meaning to λ-expressions

CENTRE DE RECHERCHE COMMUN

INRIA MICROSOFT RESEARCH

Semantics

Definition A semantics of the λ-calculus is any equivalence such that:

(1) $M \xrightarrow{\star} N$ implies $M \equiv N$

(2) $M \equiv N$ implies $C[M] \equiv C[N]$

- Thus β-interconvertibility $=_\beta$ is a semantics.
- Any other interesting semantics ?

Böhm's theorem

Theorem [Bohm, 68]

Let M and N be 2 distinct normal forms. Then for any x and y, there exists a context $C[\,]$ such that:

$$C[M] \xrightarrow{\star} x \quad \text{and} \quad C[N] \xrightarrow{\star} y$$

Corollary Any (consistent) semantics of the λ-calculus cannot identify 2 distinct normal forms.

Notice Distinct normal forms means not η-interconvertible.

Exercice Bohm's thm for $I = \lambda x.x$ and $K = \lambda x.\lambda y.x$.

Terms without normal forms

Lemma It is inconsistent to identify all terms without normal forms

Proof: Take $M = xa\Omega,\ N = y\Omega b$ where $\Omega = (\lambda x.xx)(\lambda x.xx)$

Let $C[\,] = (\lambda x.\lambda y.[\,])K(KI)$

Then $C[M] \xrightarrow{\star} a$ and $C[N] \xrightarrow{\star} b$

Question Which terms can be consistently identified ?

Easy terms `[Bohm, Jacopini]` $I = \Omega$ is consistent !

Terms without normal forms

Definition [Wadsworth, 72] M is totally undefined iff

for all $C[\,]$, if $C[M] \xrightarrow{*} nf$, then $C[N] \xrightarrow{*} nf$ for any N.

Fact: Ω is totally undefined.
$xa\Omega$ and $y\Omega b$ are not totally undefined.

Exercice:
Find other terms totally undefined. Try with $\Delta_3 = \lambda x.xxx$, $K = \lambda x.\lambda y.x$ and $Y = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$.

Terms without normal forms

Definition [Wadsworth, 72] M is in head normal form (hnf) iff

$$M = \lambda x_1.\lambda x_2.\ldots\lambda x_m.xM_1M_2\ldots M_n \quad (m, n \geq 0)$$

head variable

M not in hnf iff

$$M = \lambda x_1.\lambda x_2.\ldots\lambda x_m.(\lambda x.P)QM_1M_2\ldots M_n \quad (m, n \geq 0)$$

head redex

Proposition: M totally undefined iff M has no hnf.

Bohm trees (1/3)

Definition [72] The Bohm tree BT(M) of M is defined(?) as follows:

(1) If M has no hnf, $\mathrm{BT}(M) = \bot$

(2) If $M \xrightarrow{\star} \lambda x_1.\lambda x_2 \ldots \lambda x_m.xM_1 M_2 \ldots M_n$, then

$$\mathrm{BT}(M) = \lambda x_1.\lambda x_2 \ldots \lambda x_m.x$$

$$\mathrm{BT}(M_1) \quad \mathrm{BT}(M_2) \quad \cdots \quad \mathrm{BT}(M_n)$$

Exercices Compute BT(I), BT(K), BT(Ω), BT(Y), . . .

BT (L L L L L L L L L L L L L L L L L L L L L L L L L L L) where
L = λabcdefghijklmnopqstuvwxyzr. (r (t h i s i s a f i x e d p o i n t c o m b i n a t o r))

Bohm trees (2/3)

Theorem [74] Let $M \equiv_{\mathrm{BT}} N$ iff $\mathrm{BT}(M) = \mathrm{BT}(N)$. Then $\equiv_{\mathrm{BT}}$ is a (consistent) semantics of the λ-calculus.

Proof: (1) $M \xrightarrow{\star} N$ implies $\mathrm{BT}(M) = \mathrm{BT}(N)$.

by Church-Rosser.

(2) $\mathrm{BT}(M) = \mathrm{BT}(N)$ implies $\mathrm{BT}(C[M]) = \mathrm{BT}(C[N])$.

by completeness of inside-out reductions.

Bohm trees (3/3)

Facts [74] All Scott's semantics are quotients of equality of Bohm trees: D_∞, $P\omega$, T^ω, filter models, Jim Morris' extensional equiv.