Reductions and Causality (III)

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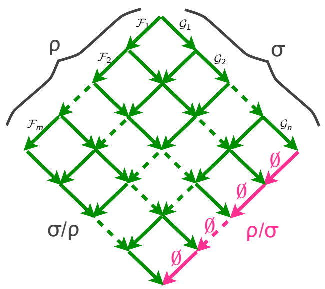
Plan

- recap
- prefix ordering
- properties of prefix ordering
- the lattice of reductions
- standard reductions as canonical reductions

Prefix ordering (1/4)

- **Definition:**
  Let ρ and σ be 2 coinitial reductions. Then ρ is prefix of σ up to permutations, $\rho \sqsubseteq \sigma$, iff $\rho/\sigma = \emptyset^m$



- Notice that $\rho \sqsubseteq \sigma$ means that $\rho \sqcup \sigma \simeq \sigma$

Properties of prefix ordering (1/3)

- **Proposition**

  (a) $\rho \sqsubseteq \sigma \sqsubseteq \rho$ iff $\rho \simeq \sigma$

  (b) $\sqsubseteq$ is an ordering relation

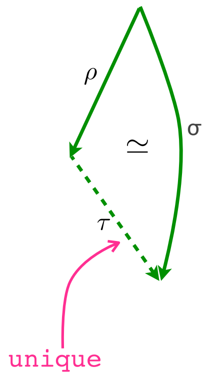
  (c) $\rho \simeq \rho' \sqsubseteq \sigma' \simeq \sigma$ implies $\rho \sqsubseteq \sigma$

  (d) $\rho \sqsubseteq \sigma$ iff $\tau\rho \sqsubseteq \tau\sigma$

  (e) $\rho \sqsubseteq \sigma$ implies $\rho/\tau \sqsubseteq \sigma/\tau$

  (f) $\rho \sqsubseteq \sigma$ iff $\exists\tau,\ \rho\tau \simeq \sigma$

  (g) $\rho \sqsubseteq \sigma$ iff $\rho \sqcup \sigma \simeq \sigma$



Properties of prefix ordering (2/3)

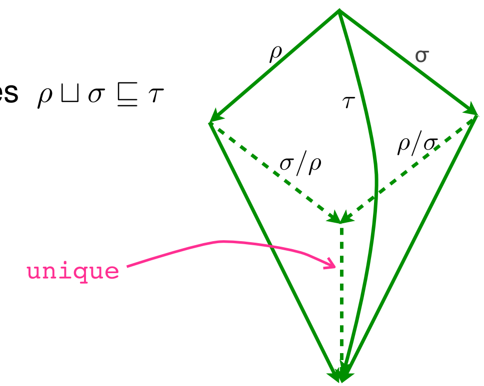
- **Proposition** `[lattice of reductions]`

  $\rho \sqsubseteq \rho \sqcup \sigma$

  $\sigma \sqsubseteq \rho \sqcup \sigma$

  $\rho \sqsubseteq \tau,\ \sigma \sqsubseteq \tau$ implies $\rho \sqcup \sigma \sqsubseteq \tau$

also named a *push-out*



Properties of prefix ordering (3/3)

- **Proposition** `[lattice of reductions]`

  $\rho \sqsubseteq \rho \sqcup \sigma$

  $\sigma \sqsubseteq \rho \sqcup \sigma$

  $\rho \sqsubseteq \tau,\ \sigma \sqsubseteq \tau$ implies $\rho \sqcup \sigma \sqsubseteq \tau$

- **Proof** First two, already proved.

Let $\rho \sqsubseteq \tau,\ \sigma \sqsubseteq \tau$. Then

$$
\begin{aligned}
&(\rho \sqcup \sigma)/\tau \\
&= (\rho/\tau)\,((\sigma/\rho)/(\tau/\rho)) \\
&= \emptyset^m\,\sigma/(\rho \sqcup \tau) \\
&= \emptyset^m\,\sigma/(\tau \sqcup \rho) \\
&= \emptyset^m\,(\sigma/\tau)/\ldots \\
&= \emptyset^m\,\emptyset^n/\ldots = \emptyset^m\,\emptyset^n
\end{aligned}
$$

Standard reductions (1/6)

- When R is a single redex, we write freely $R/\mathcal{F}$ for $\{R\}/\mathcal{F}$ or $\mathcal{F}/R$ for $\mathcal{F}/\{R\}$.

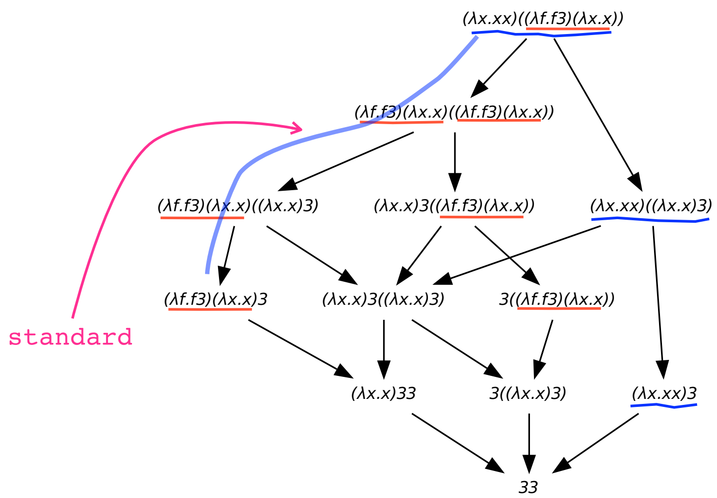
- **Proposition:**
  Let R be a redex to the left of $\mathcal{F}$. Then $R/\mathcal{F}$ is a singleton.

- **Definition:** The following reduction is **standard**

$$\rho : M = M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} M_2 \cdots \xrightarrow{R_n} M_n = N$$

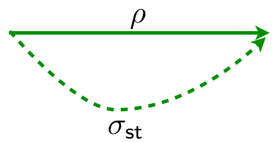
iff for all i and j, $i < j$, then R_j is not residual along ρ of some R'_j to the left of R_i in M_{i-1}.

Standard reductions (2/6)



Standard reductions (3/6)

- **Standardization thm** [Curry 50]
  Let $M \xrightarrow{\star} N$. Then $M \xrightarrow[\text{st}]{\star} N$.

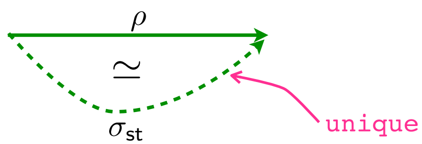


Any reduction can be performed outside-in and left-to-right.

Standard reductions (4/6)

- **Standardization thm +**
  Any ρ has a unique σ standard equivalent by permutations.

  $\forall \rho, \ \exists! \sigma_{\text{st}}, \ \rho \simeq \sigma_{\text{st}}$



Standard reductions are canonical representatives in their equivalence class by permutations.

Standard reductions (5/6)

- **Lemma** (left-to-right creation) [O'Donnell]
  Let R be redex to the left of redex S in M. Let $M \xrightarrow{S} N$.
  If T' is redex in N to the left of the residual R' of R,
  T' is residual of a redex T in M.

$$M = \cdots \underbrace{((\lambda x. \cdots S \cdots)B)}_{R} \cdots \longrightarrow \cdots \underbrace{((\lambda x. \cdots S' \cdots)B)} \cdots = N$$

$$M = \cdots \underbrace{((\lambda x.A)(\cdots S \cdots))}_{R} \cdots \longrightarrow \cdots ((\lambda x.A)(\cdots S' \cdots)) \cdots = N$$

$$M = \cdots \underbrace{((\lambda x.A)B)}_{R} \cdots S \cdots \longrightarrow \cdots ((\lambda x.A)B) \cdots S' \cdots = N$$

One cannot create a new redex across another left one.

Standard reductions (6/6)

- **Lemma** If R to the left of R_1 and ρ is standard reduction starting with contracting R_1. Then $R/\rho \neq \emptyset$.

  Proof: application of previous lemma.

- **Proof of unicity of standard reduction in each equivalence class**
  Let ρ and σ be standard and $\rho \simeq \sigma$.
  They start with same reduction and differ at some point.
  Say that ρ is more to the left than σ. Then at that point redex R contracted by ρ has (unique) residual by σ.
  Therefore $\rho \not\simeq \sigma$.

Exercices

- Show all standard reductions in the 2 reduction graphs of beginning of this class.
- Show that all reductions to normal form are equivalent.
- Show that there is a single standard reduction to normal form. What is that reduction ?
- Find an example where there is no greatest lower bound of 2 reductions. (Hint: you should use K-terms)
- Show that there is inf-lattice of reductions in λI-calculus.
- Draw lattice of reductions of $\Delta\Delta$ ($\Delta = \lambda x.xx$).
- What are standard reductions in derivations of context-free languages ?