# Reductions and Causality (II)

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July 23, 2013

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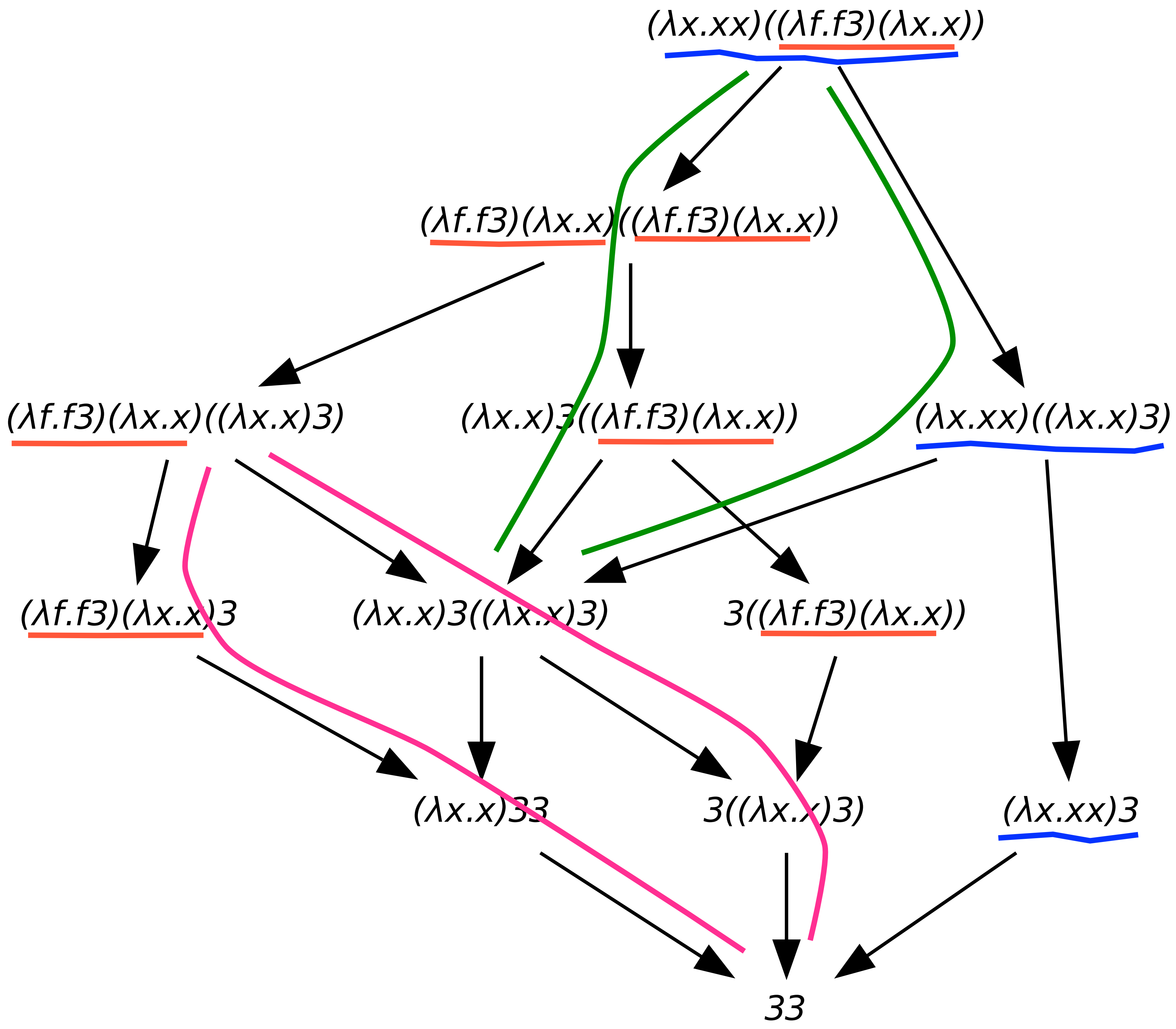
# Exercice



(λx.xx)((λf.f3)(λx.x))
(λf.f3)(λx.x)((λf.f3)(λx.x))
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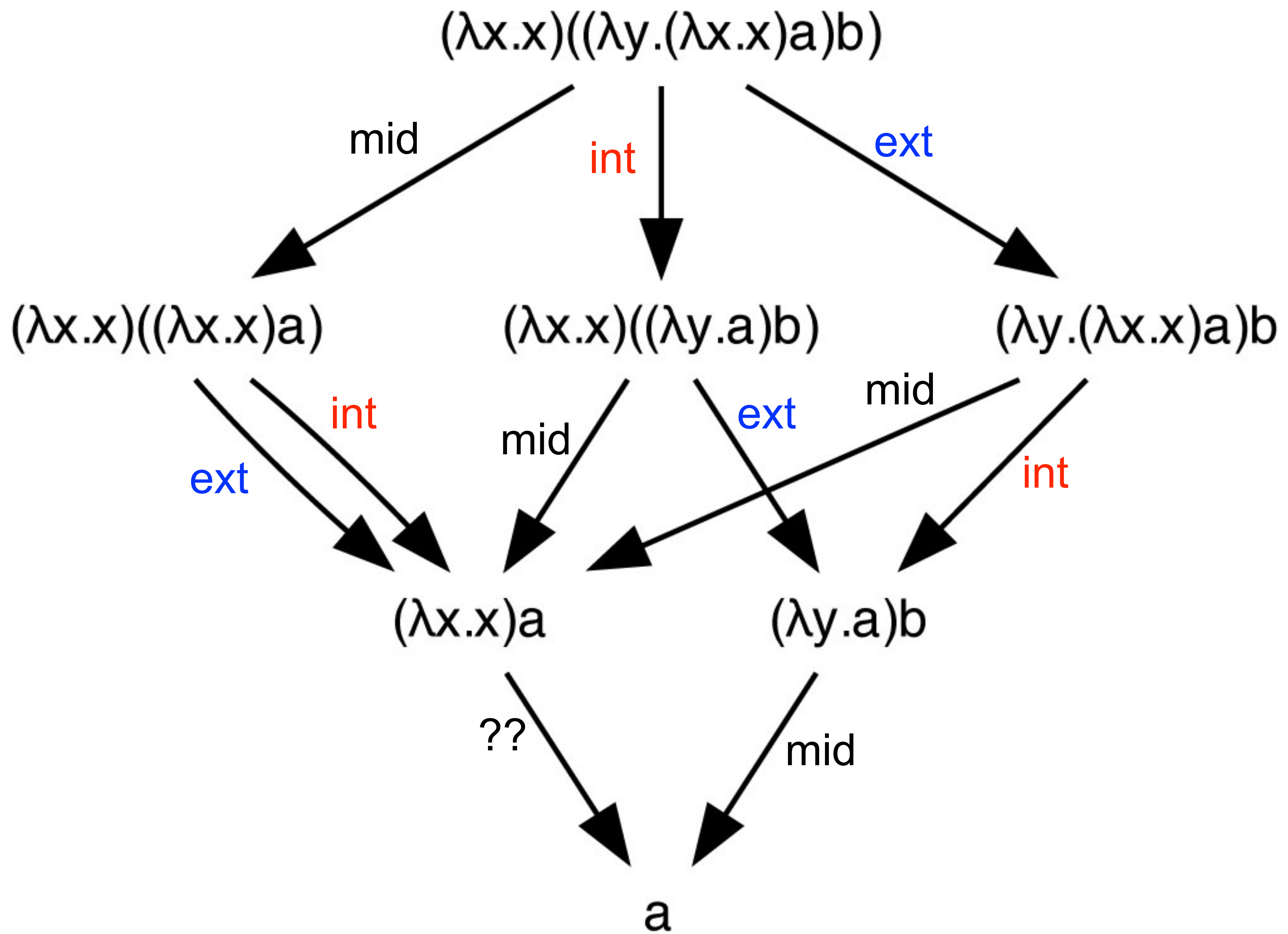
(λx.x)33
3((λx.x)3)
(λx.xx)3
33

# Exercice


(λx.x)((λy.(λx.x)a)b)
mid
int
ext
(λx.x)((λx.x)a)
(λx.x)((λy.a)b)
(λy.(λx.x)a)b
int
ext
mid
ext
mid
int
(λx.x)a
(λy.a)b
??
mid
a

# Parallel reduction steps

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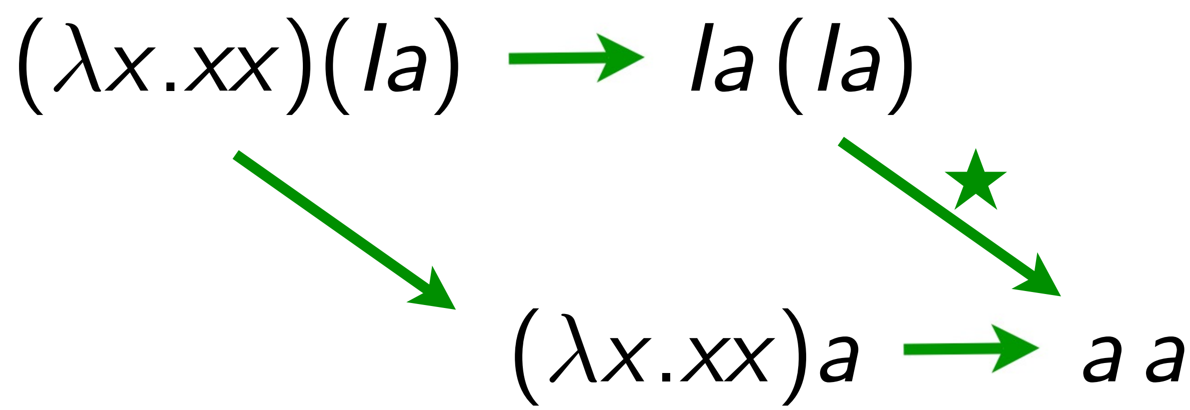
# Parallel reductions (1/3)

- permutation of reductions has to cope with copies of redexes




- in fact, a parallel reduction $Ia(Ia) \not\longrightarrow aa$

- in λ-calculus, need to define parallel reductions for nested sets

# Parallel reductions (2/3)

- the axiomatic way (à la Martin-Löf)

[Var Axiom] $x \twoheadrightarrow\!\!\!\!/\!/\; x$ [Const Axiom] $c \twoheadrightarrow\!\!\!\!/\!/\; c$

[App Rule] $$\frac{M \twoheadrightarrow\!\!\!\!/\!/\; M' \quad N \twoheadrightarrow\!\!\!\!/\!/\; N'}{MN \twoheadrightarrow\!\!\!\!/\!/\; M'N'}$$

[Abs Rule] $$\frac{M \twoheadrightarrow\!\!\!\!/\!/\; M'}{\lambda x.M \twoheadrightarrow\!\!\!\!/\!/\; \lambda x.M'}$$

[//Beta Rule] $$\frac{M \twoheadrightarrow\!\!\!\!/\!/\; M' \quad N \twoheadrightarrow\!\!\!\!/\!/\; N'}{(\lambda x.M)N \twoheadrightarrow\!\!\!\!/\!/\; M'\{x := N'\}}$$

- example:

$$(\lambda x.Ix)(Iy) \twoheadrightarrow\!\!\!\!/\!/\; (\lambda x.x)y$$

$$(\lambda x.(\lambda y.yy)x)(Ia) \twoheadrightarrow\!\!\!\!/\!/\; Ia(Ia)$$

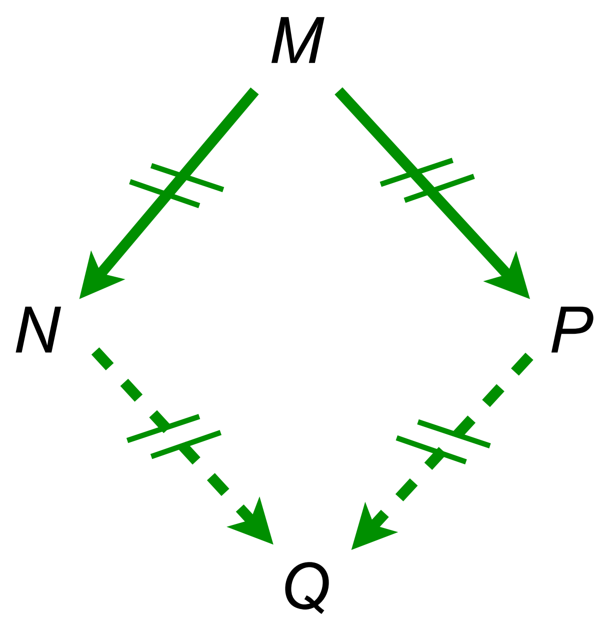
$$(\lambda x.(\lambda y.yy)x)(Ia) \twoheadrightarrow\!\!\!\!/\!/\; (\lambda y.yy)a$$

- it's an *inside-out* parallel reduction-strategy

# Parallel reductions (3/3)

- **Parallel moves lemma** [Curry 50]

If $M \twoheadrightarrow\!\!\!\!\!\!/\!\!/\;\; N$ and $M \twoheadrightarrow\!\!\!\!\!\!/\!\!/\;\; P$, then $N \twoheadrightarrow\!\!\!\!\!\!/\!\!/\;\; Q$ and $P \twoheadrightarrow\!\!\!\!\!\!/\!\!/\;\; Q$ for some $Q$.




**lemma 1-1-1-1**
**(strong confluency)**

- Enough to prove Church Rosser thm since $\longrightarrow \;\subset\; \twoheadrightarrow\!\!\!\!\!\!/\!\!/\;\; \subset\; \xrightarrow{\star}$

[Tait--Martin Löf 60?]

# Reduction of set of redexes (1/4)

- Goal: parallel reduction of a given set of redexes

$$M, N ::= x \mid \lambda x.M \mid MN \mid (\lambda x.M)^a N$$

$$a, b, c, \dots ::= \text{redex labels}$$

$$(\lambda x.M)N \longrightarrow M\{x := N\}$$

$$(\lambda x.M)^a N \longrightarrow M\{x := N\}$$

- Substitution as before with add-on:

$$((\lambda y.P)^a Q)\{x := N\} = (\lambda y.P\{x := N\})^a Q\{x := N\}$$

# Reduction of set of redexes (2/4)

- let $\mathcal{F}$ be a set of redex labels in $M$

[Var Axiom] $x \xrightarrow{\mathcal{F}} x$

[Const Axiom] $c \xrightarrow{\mathcal{F}} c$

[App Rule] $$\frac{M \xrightarrow{\mathcal{F}} M' \quad N \xrightarrow{\mathcal{F}} N'}{MN \xrightarrow{\mathcal{F}} M'N'}$$

[Abs Rule] $$\frac{M \xrightarrow{\mathcal{F}} M'}{\lambda x.M \xrightarrow{\mathcal{F}} \lambda x.M'}$$

[//Beta Rule] $$\frac{M \xrightarrow{\mathcal{F}} M' \quad N \xrightarrow{\mathcal{F}} N' \quad a \in \mathcal{F}}{(\lambda x.M)^a N \xrightarrow{\mathcal{F}} M'\{x := N'\}}$$

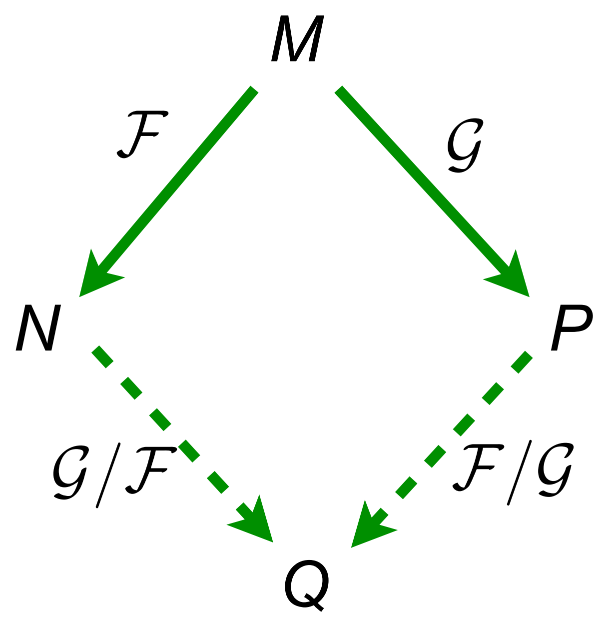
[Redex'] $$\frac{M \xrightarrow{\mathcal{F}} M' \quad N \xrightarrow{\mathcal{F}} N' \quad a \notin \mathcal{F}}{(\lambda x.M)^a N \xrightarrow{\mathcal{F}} (\lambda x.M')^a N'}$$

- let $\mathcal{F}$, $\mathcal{G}$ be set of redexes in $M$ and let $M \xrightarrow{\mathcal{F}} N$, then the set $\mathcal{G}/\mathcal{F}$ of residuals of $\mathcal{G}$ by $\mathcal{F}$ is the set of $\mathcal{G}$ redexes in $N$.

# Reduction of set of redexes (3/4)

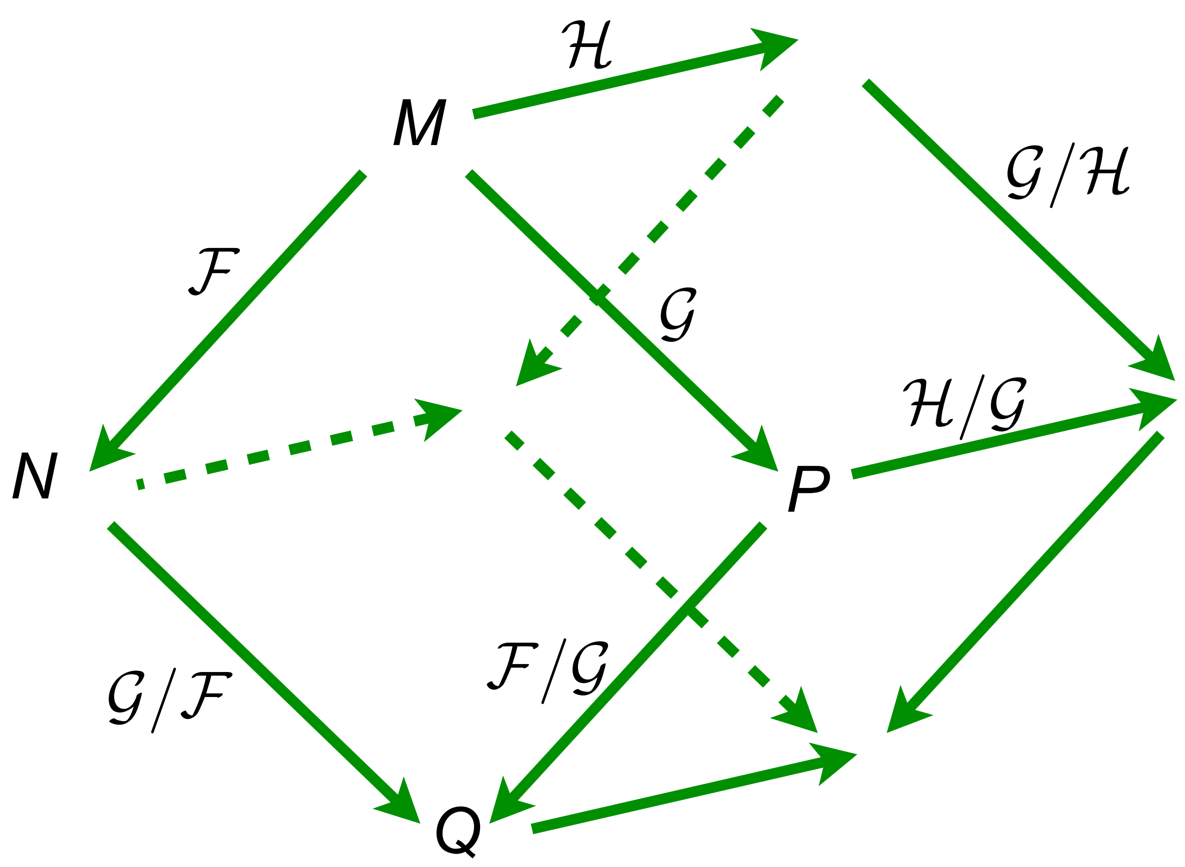
- **Parallel moves lemma+** [Curry 50]

If $M \xrightarrow{\mathcal{F}} N$ and $M \xrightarrow{\mathcal{G}} P$, then $N \xrightarrow{\mathcal{G}/\mathcal{F}} Q$ and $P \xrightarrow{\mathcal{F}/\mathcal{G}} Q$ for some $Q$.



# Reduction of set of redexes (4/4)

- **Parallel moves lemma++** [Curry 50] The Cube Lemma




- Then $(\mathcal{H}/\mathcal{F})/(\mathcal{G}/\mathcal{F}) = (\mathcal{H}/\mathcal{G})/(\mathcal{F}/\mathcal{G})$

# Recap

- WMM as an example of events causally-related
- independent and causally-related computation steps
- lemma of parallel moves
- Church-Rosser theorem
- cube lemma

# Residuals of redexes

# Redexes

- a redex is any reductible expression: $(\lambda x.M)N$

- a reduction step contracts a given redex $R = (\lambda x.A)B$

and is written: $M \xrightarrow{R} N$

- a reduction step contracts a singleton set of redexes $M \xrightarrow{\{R\}} N$

- a more precise notation would be with occurences of subterms.

We avoid it here (but it is sometimes mandatory to avoid ambiguity)

- we replaced occurences by giving names (labels) to redexes.

# Residuals of redexes (1/4)

- residuals of redexes were defined by considering labels

- they are redexes with same names when giving distinct names to initial redexes.

- a closer look w.r.t. their relative positions give following cases:

  let $R = (\lambda x.A)B$, let $M \xrightarrow{R} N$ and $S = (\lambda y.C)D$ be an other redex in $M$. Then:

# Residuals of redexes (2/4)

**Case 1:**

$$M = \cdots\cdots R \cdots\cdots \underbrace{S}\cdots\cdots \xrightarrow{R} \cdots\cdots R' \cdots\cdots \underbrace{S}\cdots\cdots = N$$

or

$$M = \cdots\cdots \underbrace{S}\cdots\cdots R \cdots\cdots \xrightarrow{R} \cdots\cdots \underbrace{S}\cdots\cdots R' \cdots\cdots = N$$

**Case 2:**

$$M = \cdots\cdots \underbrace{R}\cdots\cdots \xrightarrow{R} \cdots\cdots R' \cdots\cdots = N \quad (R \text{ and } S \text{ coincide})$$

**Case 3:**

$$M = \cdots \underbrace{(\lambda y.\cdots R \cdots)D} \cdots \xrightarrow{R} \cdots \underbrace{(\lambda y.\cdots R' \cdots)D} \cdots = N$$

**Case 4:**

$$M = \cdots \underbrace{(\lambda y.C)(\cdots R \cdots)} \cdots \xrightarrow{R} \cdots \underbrace{(\lambda y.C)(\cdots R' \cdots)} \cdots = N$$

# Residuals of redexes (3/4)

**Case 3:**

$$M = \cdots (\lambda x. \cdots \underset{\smile}{S} \cdots) B \cdots \xrightarrow{R} \cdots\cdots \underbrace{S\{x := B\}}\cdots\cdots = N$$

**Case 4:**

$$M = \cdots (\lambda x. \cdots x \cdots x \cdots)(\cdots \underset{\smile}{S} \cdots) \cdots$$

$$\xrightarrow{R} \quad \cdots\cdots(\cdots \underset{\smile}{S} \cdots)\cdots(\cdots \underset{\smile}{S} \cdots)\cdots\cdots = N$$

# Residuals of redexes (4/4)

**Examples:** $\Delta = \lambda x.xx, I = \lambda x.x$

$$\Delta(\underbrace{I\,x}) \longrightarrow \underbrace{I\,x}(\underbrace{I\,x})$$

$$\underbrace{I\,x}\,(\Delta(I\,x)) \longrightarrow \underbrace{I\,x}\,(I\,x(I\,x))$$

$$\underbrace{I\,(\Delta(I\,x))} \longrightarrow \underbrace{I\,(I\,x(I\,x))}$$

$$\underbrace{\Delta(I\,x)} \longrightarrow I\,x(I\,x)$$

$$I\,x\,(\Delta(\underbrace{I\,x})) \longrightarrow I\,x\,(\underbrace{I\,x}(\underbrace{I\,x}))$$

$$\underbrace{\Delta\Delta} \longrightarrow \Delta\Delta$$

# Residuals of reductions

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# Parallel reductions

- Redex occurences and labels
  - Let $||U|| = M$ where labels in $U$ are erased (forgetful functor)
  - Then $M \xrightarrow{\mathcal{F}} N$ iff $U \xrightarrow{\mathcal{F}} N$ for some labeled $U$ and $M = ||U||$

- Consider reductions where each step is parallel

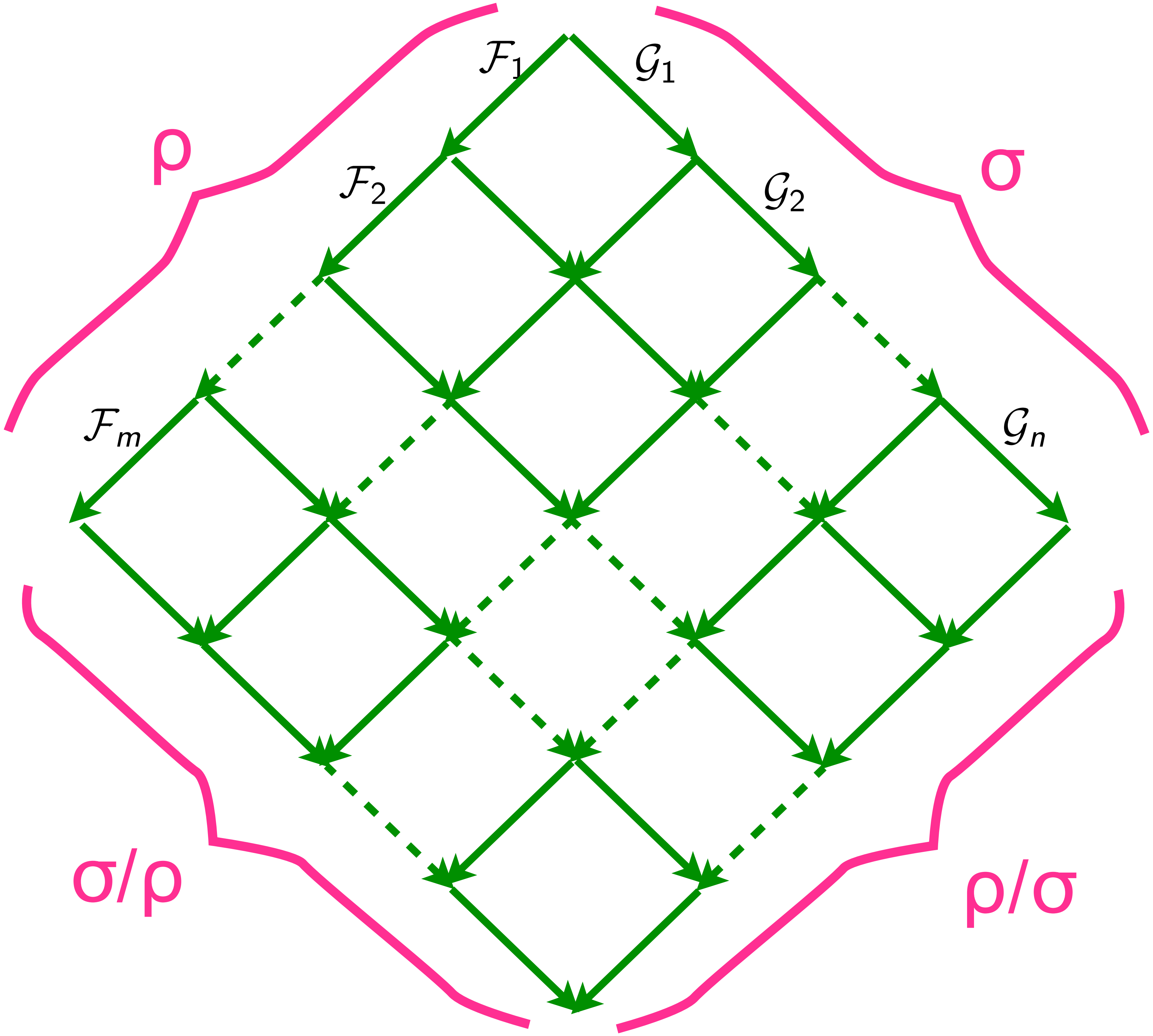
$$\rho : M = M_0 \xrightarrow{\mathcal{F}_1} M_1 \xrightarrow{\mathcal{F}_2} M_2 \cdots \xrightarrow{\mathcal{F}_n} M_n = N$$

- We also write

$$\rho = 0 \text{ when } n = 0$$

$$\rho = \mathcal{F}_1 \mathcal{F}_2 \cdots \mathcal{F}_n \text{ when } M \text{ clear from context}$$

# Residual of reduction (1/4)



# Residual of reduction (2/4)

- **Definition** [JJL 76]

  $$\rho/0 = \rho$$

  $$\rho/(\sigma\,\tau) = (\rho/\sigma)/\tau$$

  $$(\rho\,\sigma)/\tau = (\rho/\tau)\,(\sigma/(\tau/\rho))$$

  $\mathcal{F}/\mathcal{G}$ already defined

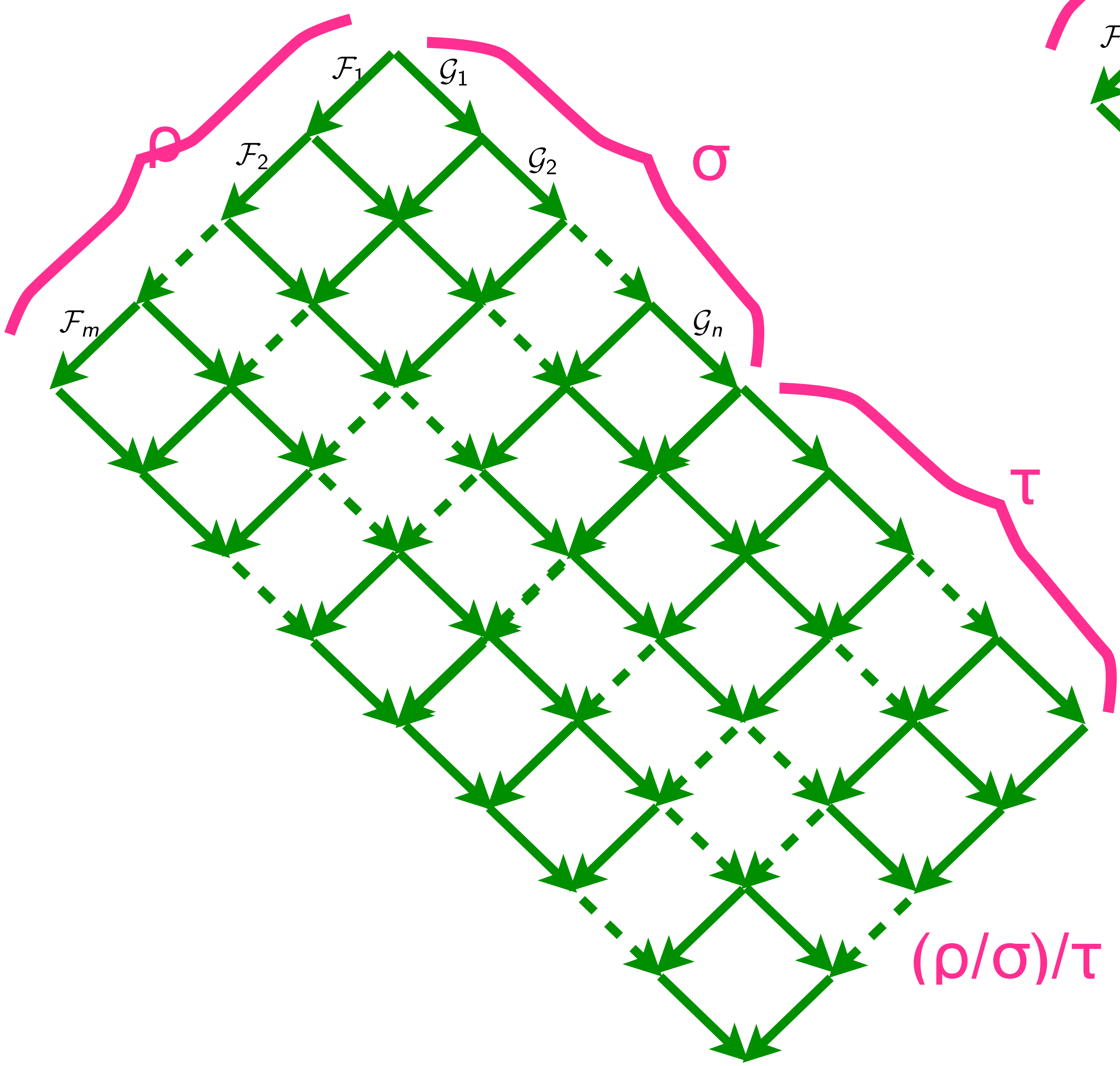
- **Notation**

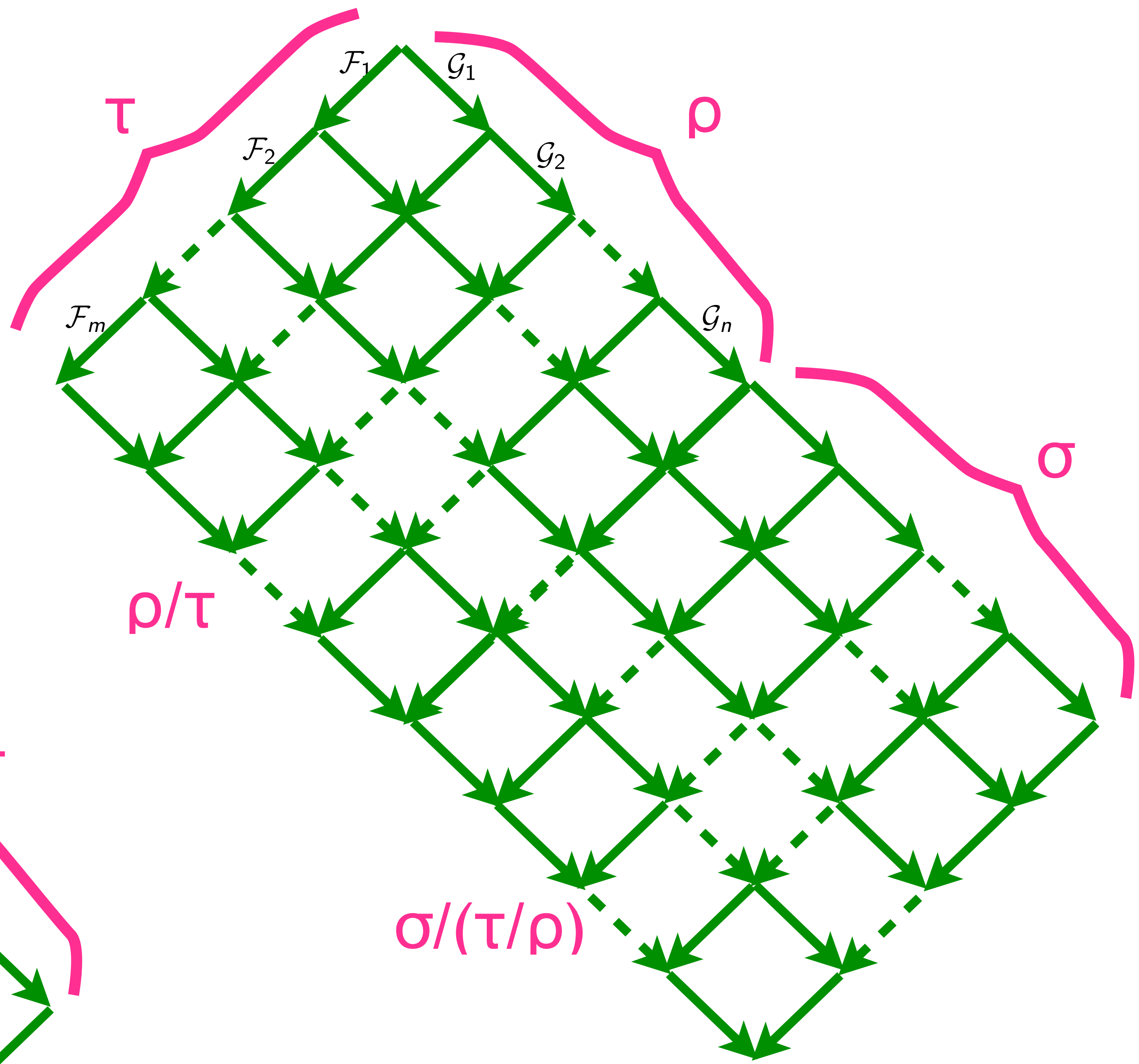
  $$\rho \sqcup \sigma = \rho\,(\sigma/\rho)$$

- **Proposition** [Parallel Moves +]:

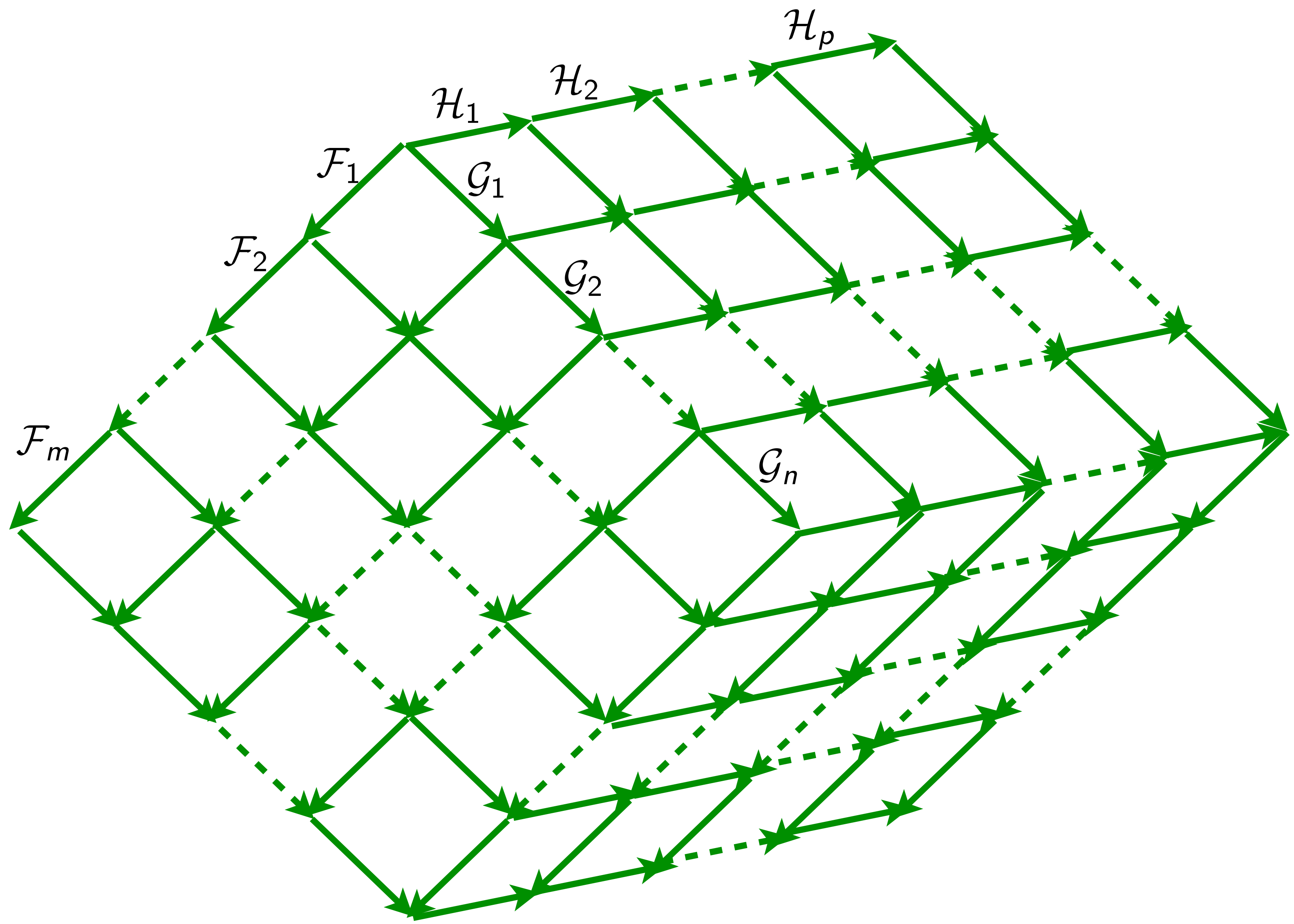
  $\rho \sqcup \sigma$ and $\sigma \sqcup \rho$ are cofinal

# Residual of reduction (3/4)



# Residual of reduction (4/4)




- **Proposition** [Cube Lemma ++]:

$$\tau/(\rho \sqcup \sigma) = \tau/(\sigma \sqcup \rho)$$

# Equivalence by permutations

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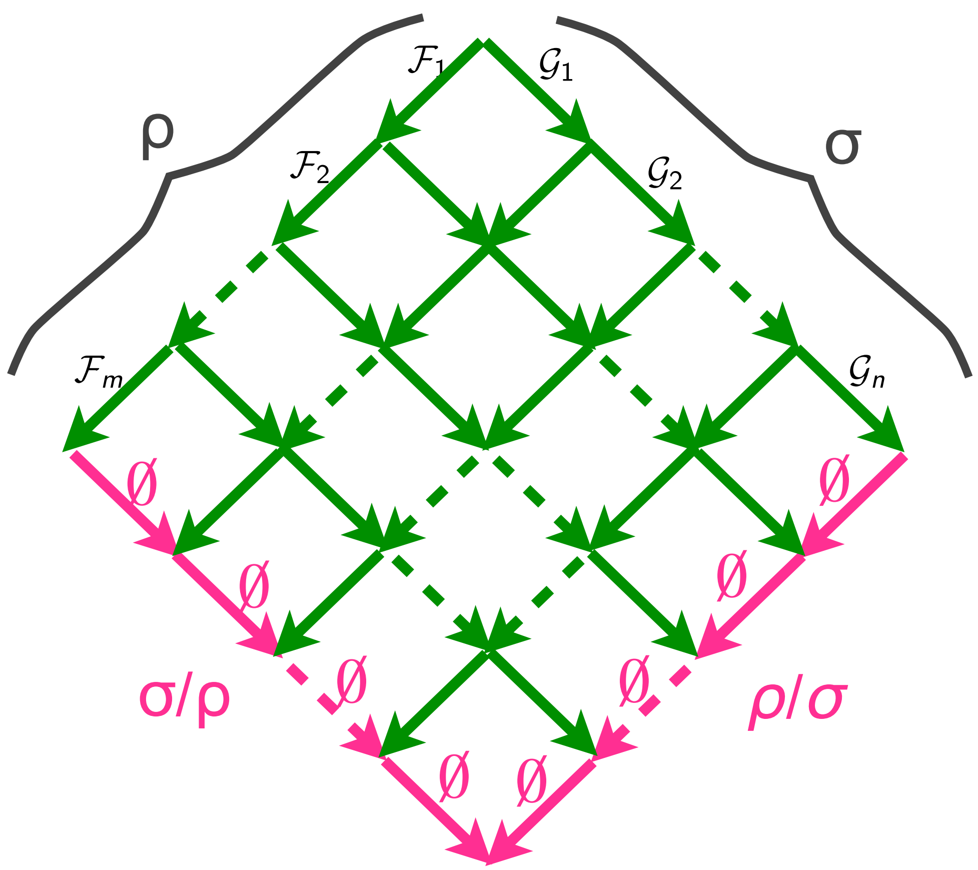
# Equivalence by permutations (1/4)

- **Definition:**

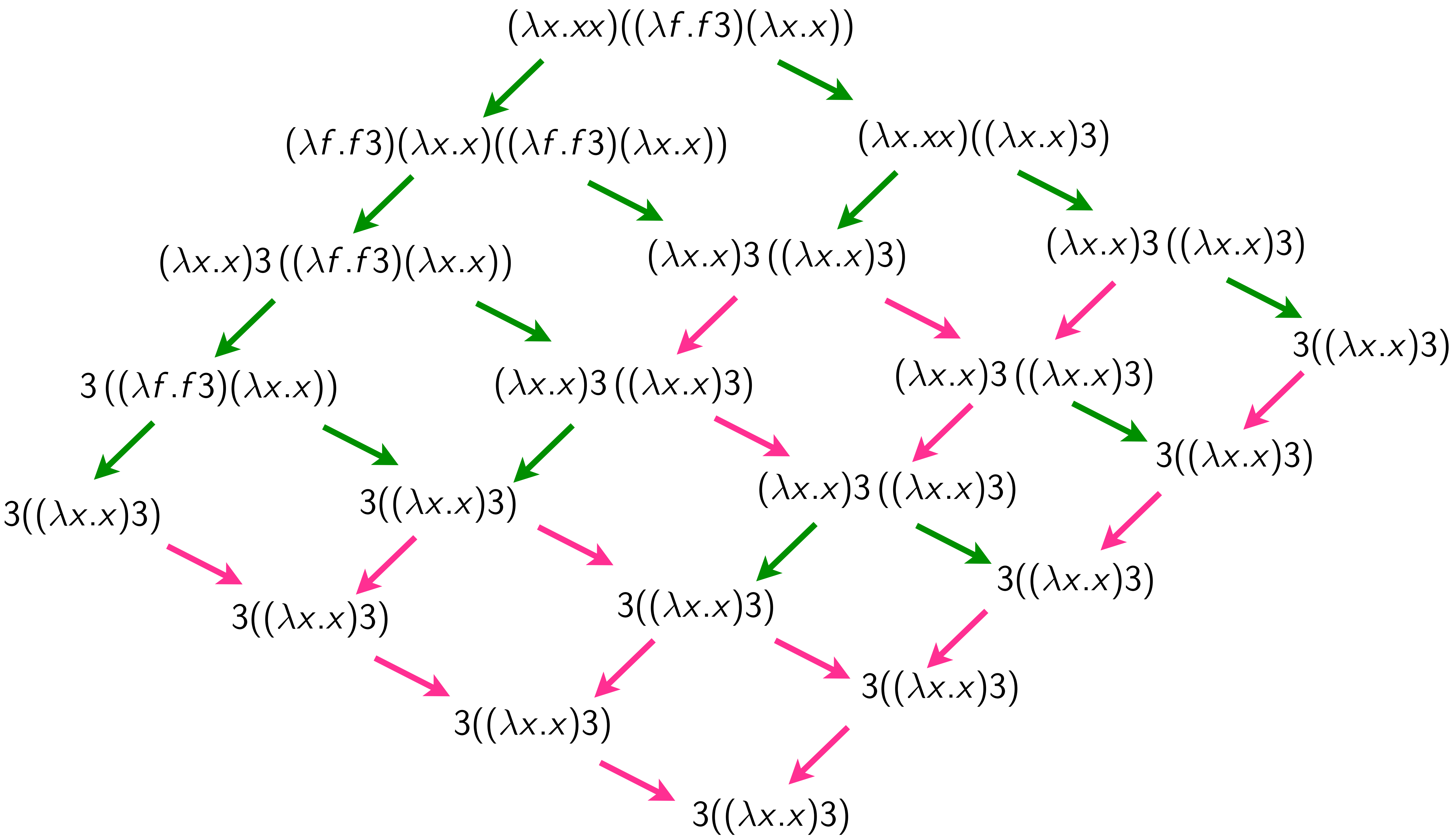
  Let $\rho$ and $\sigma$ be 2 coinitial reductions. Then $\rho$ is equivalent to $\sigma$ by permutations, $\rho \simeq \sigma$, iff:

$$\rho/\sigma = \emptyset^m \quad \text{and} \quad \sigma/\rho = \emptyset^n$$

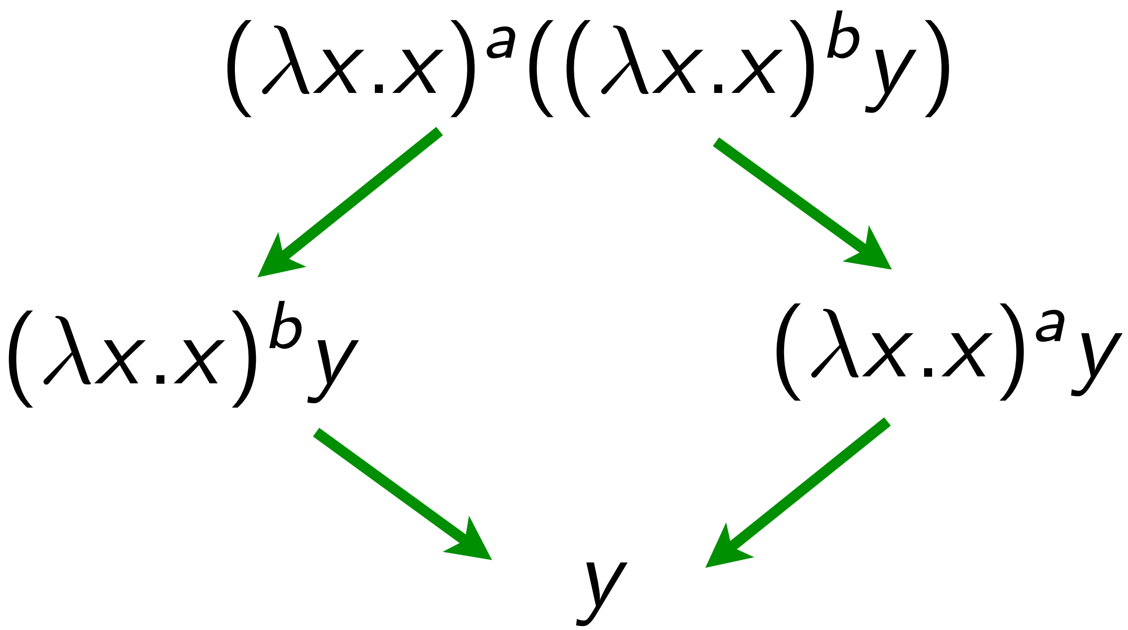



- Notice that $\rho \simeq \sigma$ means that $\rho$ and $\sigma$ are cofinal

# Equivalence by permutations (2/4)



# Equivalence by permutations (3/4)




$$\rho : M = I(Iy) \xrightarrow{R_a} Iy = N$$

$$\sigma : M = I(Iy) \xrightarrow{R_b} Iy = N$$

$$\rho \not\simeq \sigma$$

- Notice that $\rho \not\simeq \sigma$ while $\rho$ and $\sigma$ are coinitial and cofinal

# Equivalence by permutations (4/4)

- Same with $0 \not\simeq \rho$ when $\rho : \Delta\Delta \longrightarrow \Delta\Delta$
  $\Delta = \lambda x.xx$

- **Exercice 1**: Give other examples of non-equivalent reductions between same terms

- **Exercice 2**: Show following equalities

$$\rho/0 = \rho \qquad \emptyset^n/\rho = \emptyset^n$$
$$0/\rho = 0 \qquad 0 \simeq \emptyset^n$$
$$\rho/\emptyset^n = \rho \qquad \rho/\rho = \emptyset^n$$

- **Exercice 3**: Show that $\simeq$ is an equivalence relation.

# Properties of equivalent reductions

- **Proposition**

$$\rho \simeq \sigma \;\text{ iff }\; \forall\tau,\; \tau/\rho = \tau/\sigma$$

$$\rho \simeq \sigma \text{ implies } \rho/\tau \simeq \sigma/\tau$$

$$\rho \simeq \sigma \;\text{ iff }\; \tau\rho \simeq \tau\sigma$$

$$\rho \simeq \sigma \;\text{ implies }\; \rho\tau \simeq \sigma\tau$$

$$\rho \sqcup \sigma \simeq \sigma \sqcup \rho$$

- **Proof**

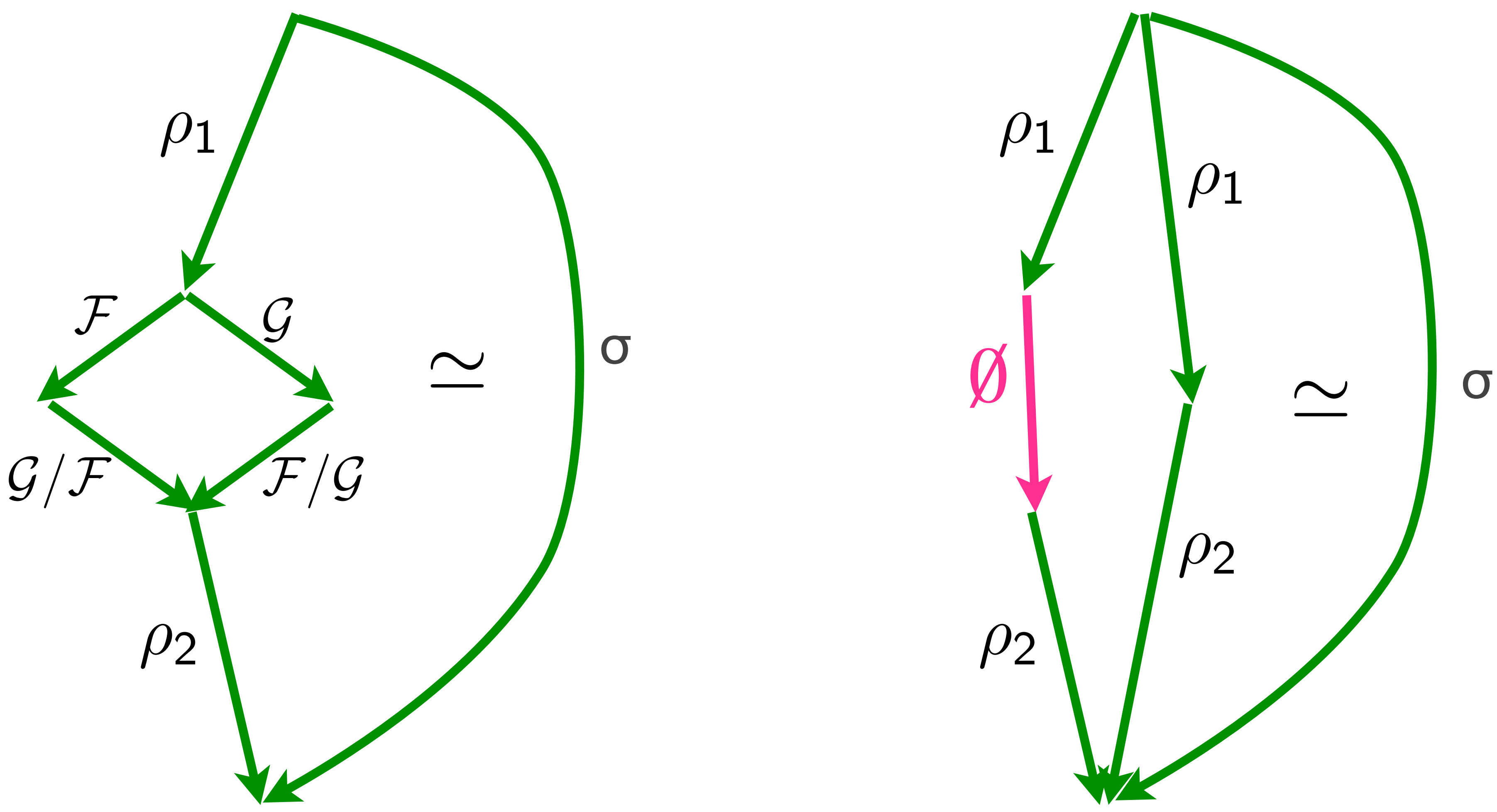
As $\rho \simeq \sigma$, one has $\sigma/\rho = \emptyset^n$. Therefore $\tau/\rho = (\tau/\rho)/(\sigma/\rho)$. That is $\tau/\rho = \tau/(\rho \sqcup \sigma)$. Similarly as $\sigma \simeq \rho$, one gets $\tau/\sigma = \tau/(\sigma \sqcup \rho)$. But cube lemma says $\tau/(\rho \sqcup \sigma) = \tau/(\sigma \sqcup \rho)$. Therefore $\tau/\rho = \tau/\sigma$.

# Properties of equivalent reductions

- **Proposition** $\simeq$ is the smallest congruence containing

$$\mathcal{F}\,(\mathcal{G}/\mathcal{F}) \simeq \mathcal{G}\,(\mathcal{F}/\mathcal{G})$$

$$0 \simeq \emptyset$$



# Beyond the λ-calculus

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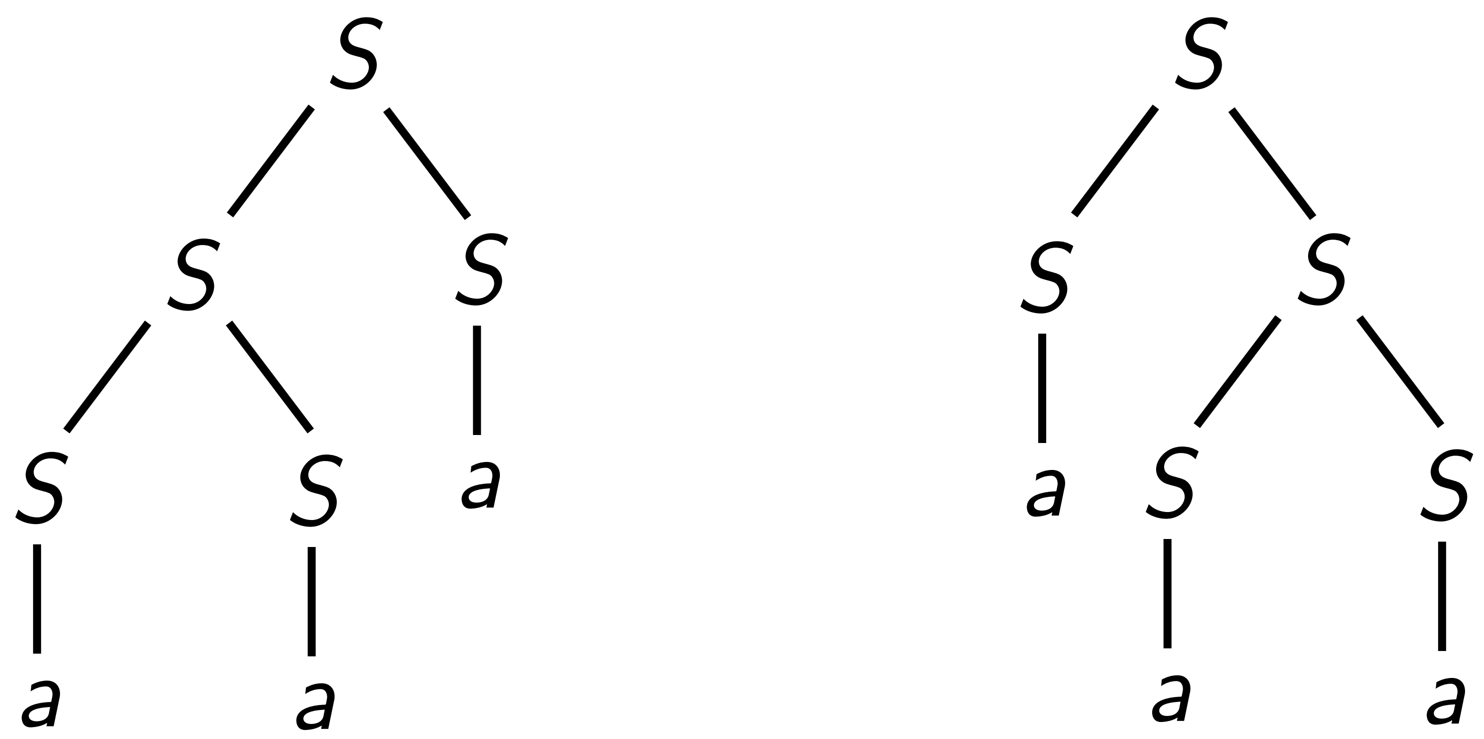
# Context-free languages

- permutations of derivations in contex-free languages

$$S \to SS$$

$$S \to a$$




- each parse tree corresponds to an equivalence class

# Term rewriting

- permutations of derivations are defined with critical pairs
- critical pairs make conflicts
- only 2nd definition of equivalence works [Boudol, 1982]

# Process algebras

- similar to TRS [Boudol-Castellani, 1982]

# Exercices

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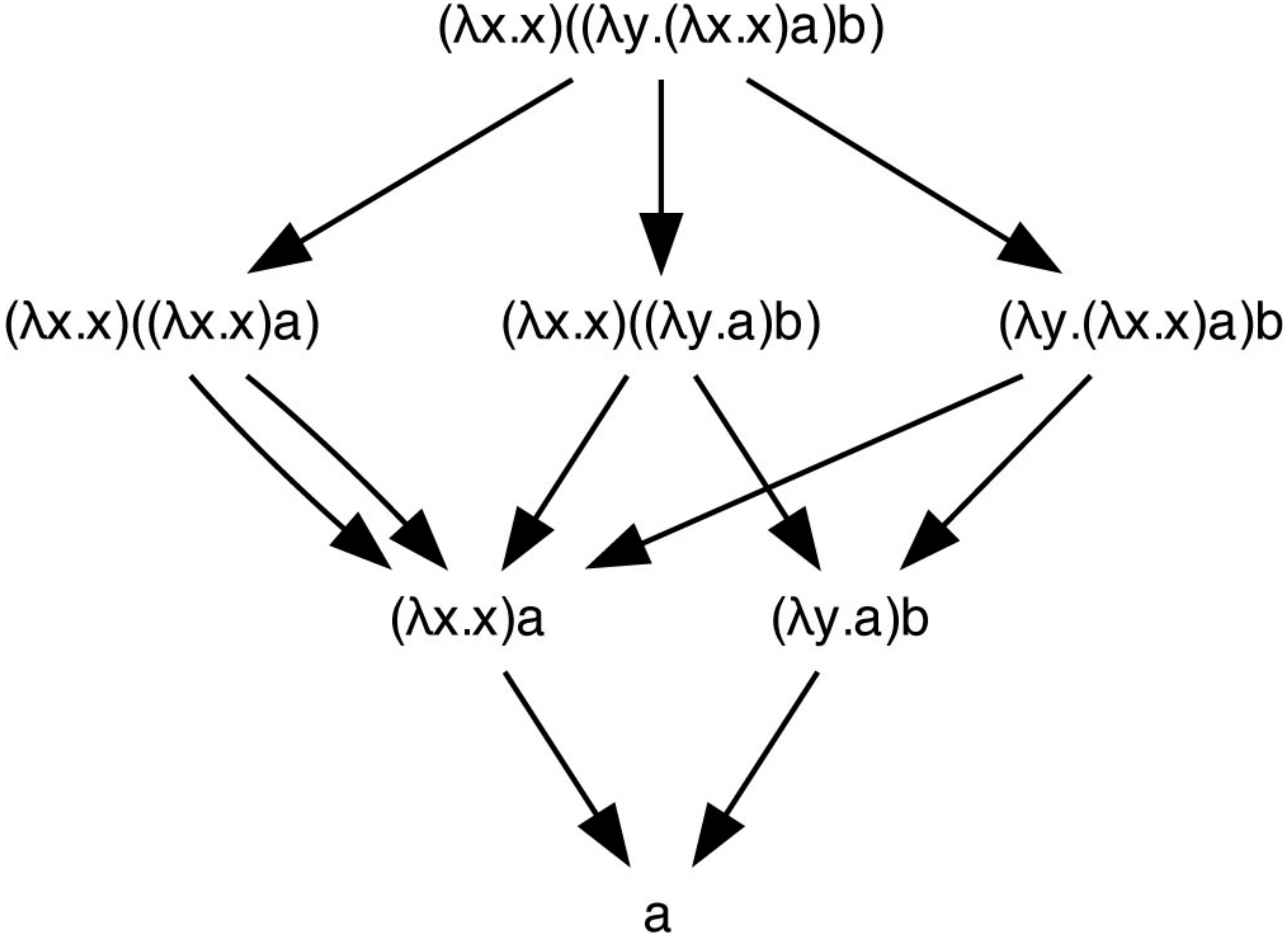
# Exercices

- **Exercice 4**: Complete all proofs of propositions
- **Exercice 5**: Show equivalent reductions in



# Proof Parallel moves

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# Parallel moves (1/4)

- Lemma $M \xrightarrow{\mathcal{F}} N,\ M \xrightarrow{\mathcal{G}} P \ \Rightarrow\ N \xrightarrow{\mathcal{F}} Q,\ P \xrightarrow{\mathcal{G}} Q$

  Proof

  Case 1: $M = x = N = P = Q$. Obvious.

  Case 2: $M = \lambda x.M_1,\ N = \lambda x.N_1,\ P = \lambda x.P_1$. Obvious by induction on $M_1$

  Case 3: (App-App) $M = M_1 M_2,\ N = N_1 N_2,\ P = P_1 P_2$. Obvious by induction on $M_1,\ M_2$.

  Case 4: (Red'-Red') $M = (\lambda x.M_1)^a M_2,\ N = (\lambda x.N_1)^a N_2,\ P = (\lambda x.P_1)^a P_2,\ a \notin \mathcal{F} \cup \mathcal{G}$
  Then induction on $M_1,\ M_2$.

  Case 4: (beta-Red') $M = (\lambda x.M_1)^a M_2,\ N = N_1\{x := N_2\},\ P = (\lambda x.P_1)^a P_2,\ a \in \mathcal{F}, a \notin \mathcal{G}$
  By induction $N_1 \xrightarrow{\mathcal{G}} Q_1,\ P_1 \xrightarrow{\mathcal{F}} Q_1$. And $N_2 \xrightarrow{\mathcal{G}} Q_2,\ P_1 \xrightarrow{\mathcal{F}} Q_2$.
  By lemma, $N_1\{x := N_2\} \xrightarrow{\mathcal{G}} Q_1\{x := Q_2\}$. And $(\lambda x.P_1)^a P_2 \xrightarrow{\mathcal{F}} Q_1\{x := Q_2\}$

  Case 5: (beta-beta) $M = (\lambda x.M_1)^a M_2,\ N = N_1\{x := N_2\},\ P = P_1\{x := P_2\},\ a \in \mathcal{F} \cap \mathcal{G}$
  As before with same lemma.

# Parallel moves (1/4)

- Lemma $M \xrightarrow{\mathcal{F}} N, P \xrightarrow{\mathcal{F}} Q \Rightarrow M\{x := P\} \xrightarrow{\mathcal{F}} N\{x := Q\}$

  Proof: exercice!

- Lemma [subst] $M\{x := N\}\{y := P\} = M\{y := P\}\{x := N\{y := P\}\}$

  when $x$ not free in $P$