Reductions and Causality (II)

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July 23, 2013

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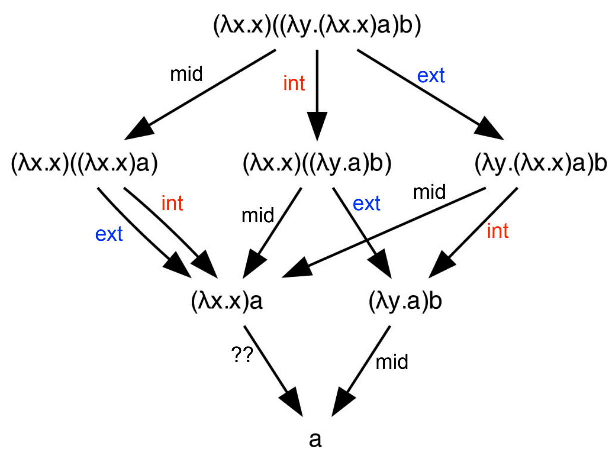
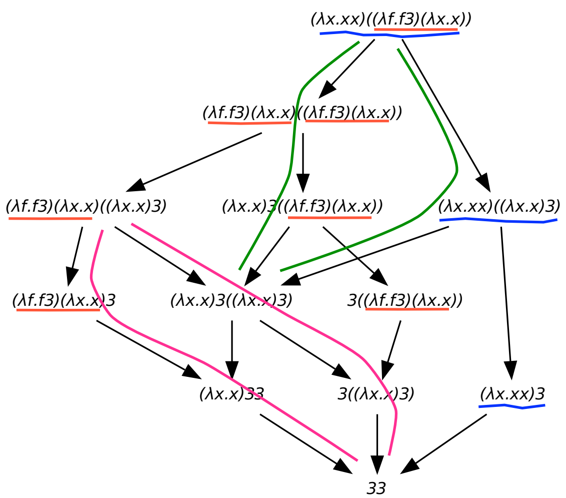
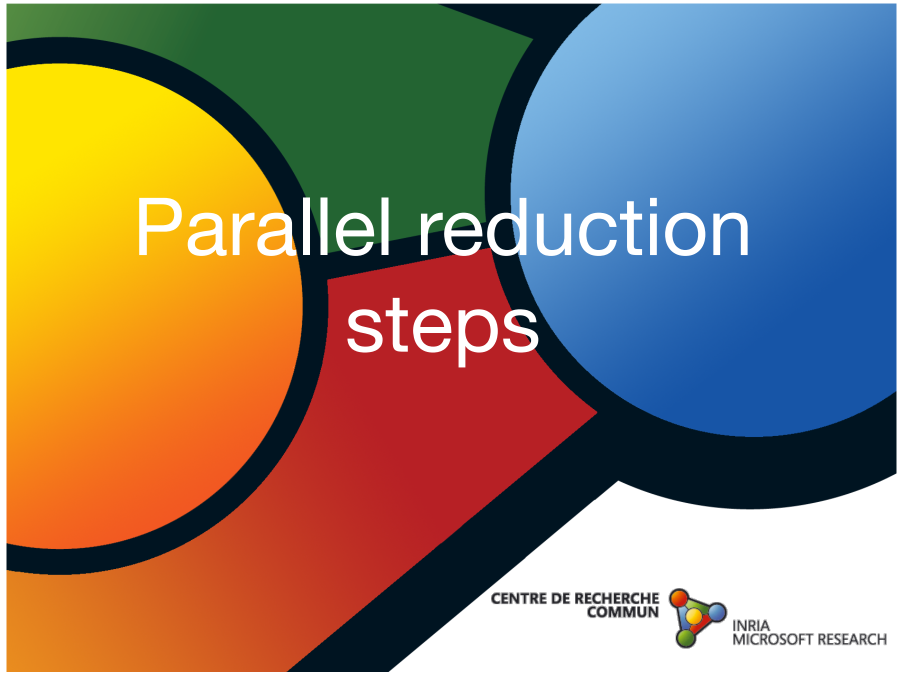
Exercice



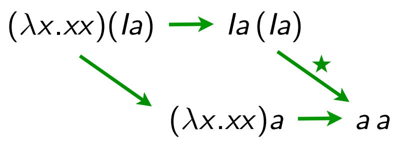
Exercice





Parallel reductions (1/3)

- permutation of reductions has to cope with copies of redexes



- in fact, a parallel reduction $Ia(Ia) \mathrel{/\!\!/\!\!\!\rightarrow} aa$
- in λ-calculus, need to define parallel reductions for nested sets

Parallel reductions (2/3)

- the axiomatic way (à la Martin-Löf)

[Var Axiom] $x \mathrel{/\!\!/\!\!\!\rightarrow} x$ [Const Axiom] $c \mathrel{/\!\!/\!\!\!\rightarrow} c$

[App Rule] $\dfrac{M \mathrel{/\!\!/\!\!\!\rightarrow} M' \quad N \mathrel{/\!\!/\!\!\!\rightarrow} N'}{MN \mathrel{/\!\!/\!\!\!\rightarrow} M'N'}$ [Abs Rule] $\dfrac{M \mathrel{/\!\!/\!\!\!\rightarrow} M'}{\lambda x.M \mathrel{/\!\!/\!\!\!\rightarrow} \lambda x.M'}$

[//Beta Rule] $\dfrac{M \mathrel{/\!\!/\!\!\!\rightarrow} M' \quad N \mathrel{/\!\!/\!\!\!\rightarrow} N'}{(\lambda x.M)N \mathrel{/\!\!/\!\!\!\rightarrow} M'\{x := N'\}}$

- example:

$$(\lambda x.Ix)(Iy) \mathrel{/\!\!/\!\!\!\rightarrow} (\lambda x.x)y$$

$$(\lambda x.(\lambda y.yy)x)(Ia) \mathrel{/\!\!/\!\!\!\rightarrow} Ia(Ia)$$

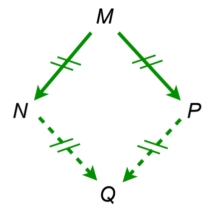
$$(\lambda x.(\lambda y.yy)x)(Ia) \mathrel{/\!\!/\!\!\!\rightarrow} (\lambda y.yy)a$$

- it's an *inside-out* parallel reduction-strategy

Parallel reductions (3/3)

- **Parallel moves lemma** `[Curry 50]`

If $M \mathrel{/\!\!/\!\!\!\rightarrow} N$ and $M \mathrel{/\!\!/\!\!\!\rightarrow} P$, then $N \mathrel{/\!\!/\!\!\!\rightarrow} Q$ and $P \mathrel{/\!\!/\!\!\!\rightarrow} Q$ for some Q.



lemma 1-1-1-1
(strong confluency)

- Enough to prove Church Rosser thm since $\rightarrow \;\subset\; \mathrel{/\!\!/\!\!\!\rightarrow} \;\subset\; \xrightarrow{\star}$

`[Tait--Martin Löf 60?]`

Reduction of set of redexes (1/4)

- Goal: parallel reduction of a given set of redexes

$$M, N ::= x \mid \lambda x.M \mid MN \mid (\lambda x.M)^a N$$

$a, b, c, \ldots ::=$ redex labels

$$(\lambda x.M)N \longrightarrow M\{x := N\}$$

$$(\lambda x.M)^a N \longrightarrow M\{x := N\}$$

- Substitution as before with add-on:

$$((\lambda y.P)^a Q)\{x := N\} = (\lambda y.P\{x := N\})^a Q\{x := N\}$$

Reduction of set of redexes (2/4)

- let $\mathcal{F}$ be a set of redex labels in M

[Var Axiom] $x \xrightarrow{\mathcal{F}} x$ [Const Axiom] $c \xrightarrow{\mathcal{F}} c$

[App Rule] $\dfrac{M \xrightarrow{\mathcal{F}} M' \quad N \xrightarrow{\mathcal{F}} N'}{MN \xrightarrow{\mathcal{F}} M'N'}$ [Abs Rule] $\dfrac{M \xrightarrow{\mathcal{F}} M'}{\lambda x.M \xrightarrow{\mathcal{F}} \lambda x.M'}$

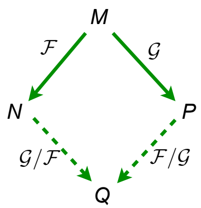
[//Beta Rule] $\dfrac{M \xrightarrow{\mathcal{F}} M' \quad N \xrightarrow{\mathcal{F}} N' \quad a \in \mathcal{F}}{(\lambda x.M)^a N \xrightarrow{\mathcal{F}} M'\{x := N'\}}$ [Redex'] $\dfrac{M \xrightarrow{\mathcal{F}} M' \quad N \xrightarrow{\mathcal{F}} N' \quad a \notin \mathcal{F}}{(\lambda x.M)^a N \xrightarrow{\mathcal{F}} (\lambda x.M')^a N'}$

- let $\mathcal{F}$, $\mathcal{G}$ be set of redexes in M and let $M \xrightarrow{\mathcal{F}} N$, then the set $\mathcal{G}/\mathcal{F}$ of residuals of $\mathcal{G}$ by $\mathcal{F}$ is the set of $\mathcal{G}$ redexes in N.

Reduction of set of redexes (3/4)

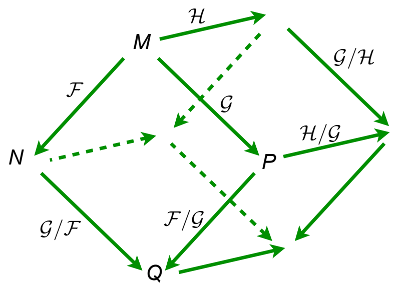
- **Parallel moves lemma+** [Curry 50]

If $M \xrightarrow{\mathcal{F}} N$ and $M \xrightarrow{\mathcal{G}} P$, then $N \xrightarrow{\mathcal{G}/\mathcal{F}} Q$ and $P \xrightarrow{\mathcal{F}/\mathcal{G}} Q$ for some Q.



Reduction of set of redexes (4/4)

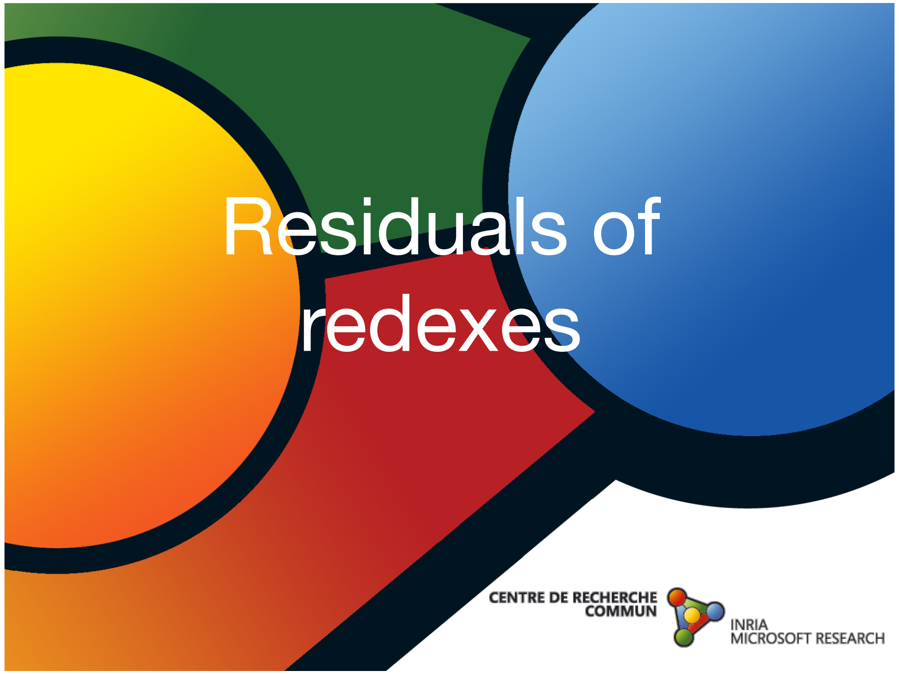
- **Parallel moves lemma++** [Curry 50] The Cube Lemma



- Then $(\mathcal{H}/\mathcal{F})/(\mathcal{G}/\mathcal{F}) = (\mathcal{H}/\mathcal{G})/(\mathcal{F}/\mathcal{G})$

Recap

- WMM as an example of events causally-related
- independent and causally-related computation steps
- lemma of parallel moves
- Church-Rosser theorem
- cube lemma



Residuals of redexes (1/4)

- residuals of redexes were defined by considering labels
- they are redexes with same names when giving distinct names to initial redexes.
- a closer look w.r.t. their relative positions give following cases:

let $R = (\lambda x.A)B$, let $M \xrightarrow{R} N$ and $S = (\lambda y.C)D$ be an other redex in M. Then:

Redexes

- a redex is any reductible expression: $(\lambda x.M)N$
- a reduction step contracts a given redex $R = (\lambda x.A)B$ and is written: $M \xrightarrow{R} N$
- a reduction step contracts a singleton set of redexes $M \xrightarrow{\{R\}} N$
- a more precise notation would be with occurences of subterms. We avoid it here (but it is sometimes mandatory to avoid ambiguity)
- we replaced occurences by giving names (labels) to redexes.

Residuals of redexes (2/4)

Case 1:

$$M = \cdots\cdots R \cdots\cdots \underline{S} \cdots\cdots \xrightarrow{R} \cdots\cdots R' \cdots\cdots \underline{S} \cdots\cdots = N$$

or

$$M = \cdots\cdots \underline{S} \cdots\cdots R \cdots\cdots \xrightarrow{R} \cdots\cdots \underline{S} \cdots\cdots R' \cdots\cdots = N$$

Case 2:

$$M = \cdots\cdots \underline{R} \cdots\cdots \xrightarrow{R} \cdots\cdots R' \cdots\cdots = N \quad (R \text{ and } S \text{ coincide})$$

Case 3:

$$M = \cdots \underline{(\lambda y. \cdots R \cdots)D} \cdots \xrightarrow{R} \cdots \underline{(\lambda y. \cdots R' \cdots)D} \cdots = N$$

Case 4:

$$M = \cdots \underline{(\lambda y.C)(\cdots R \cdots)} \cdots \xrightarrow{R} \cdots \underline{(\lambda y.C)(\cdots R' \cdots)} \cdots = N$$

Residuals of redexes (3/4)

Case 3:

$$M = \cdots(\lambda x.\cdots S \cdots)B \cdots \xrightarrow{R} \cdots\cdots S\{x := B\} \cdots\cdots = N$$

Case 4:

$$M = \cdots(\lambda x.\cdots x \cdots x \cdots)(\cdots S \cdots)\cdots$$

$$\searrow^{R} \quad \cdots\cdots(\cdots S \cdots)\cdots(\cdots S \cdots)\cdots\cdots = N$$

Residuals of reductions

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Residuals of redexes (4/4)

Examples: $\Delta = \lambda x.xx,\ I = \lambda x.x$

$$\Delta(I\,x) \longrightarrow I\,x(I\,x)$$

$$I\,x\,(\Delta(I\,x)) \longrightarrow I\,x\,(I\,x(I\,x))$$

$$I\,(\Delta(I\,x)) \longrightarrow I\,(I\,x(I\,x))$$

$$\Delta(I\,x) \longrightarrow I\,x(I\,x)$$

$$I\,x\,(\Delta(I\,x)) \longrightarrow I\,x\,(I\,x(I\,x))$$

$$\Delta\Delta \longrightarrow \Delta\Delta$$

Parallel reductions

- Redex occurences and labels
  - Let $||U|| = M$ where labels in U are erased (forgetful functor)
  - Then $M \xrightarrow{\mathcal{F}} N$ iff $U \xrightarrow{\mathcal{F}} N$ for some labeled U and $M = ||U||$
- Consider reductions where each step is parallel

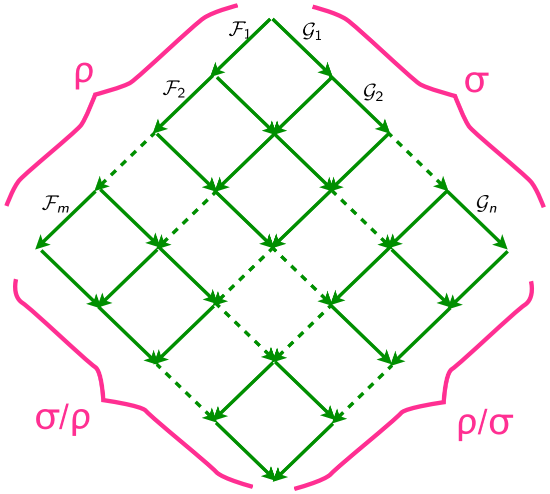
$$\rho : M = M_0 \xrightarrow{\mathcal{F}_1} M_1 \xrightarrow{\mathcal{F}_2} M_2 \cdots \xrightarrow{\mathcal{F}_n} M_n = N$$

- We also write

$\rho = 0$ when $n = 0$

$\rho = \mathcal{F}_1\,\mathcal{F}_2 \cdots \mathcal{F}_n$ when M clear from context

Residual of reduction (1/4)



Residual of reduction (2/4)

- **Definition** [JJL 76]

  $\rho/0 = \rho$

  $\rho/(\sigma\,\tau) = (\rho/\sigma)/\tau$

  $(\rho\,\sigma)/\tau = (\rho/\tau)\,(\sigma/(\tau/\rho))$

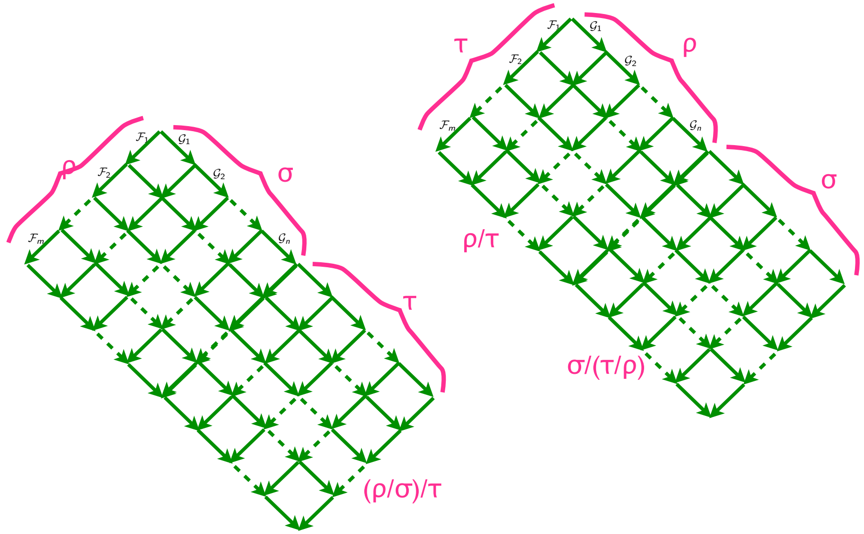
  $\mathcal{F}/\mathcal{G}$ already defined

- **Notation**

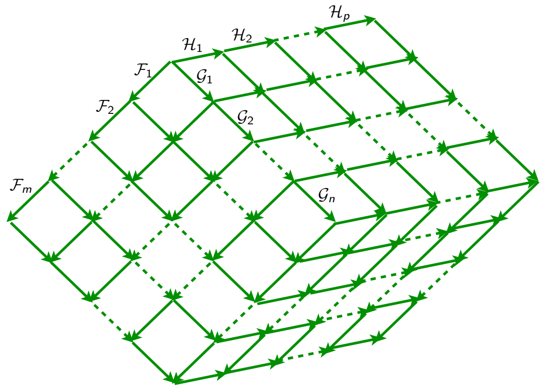
  $\rho \sqcup \sigma = \rho\,(\sigma/\rho)$

- **Proposition** [Parallel Moves +]:
  $\rho \sqcup \sigma$ and $\sigma \sqcup \rho$ are cofinal

Residual of reduction (3/4)

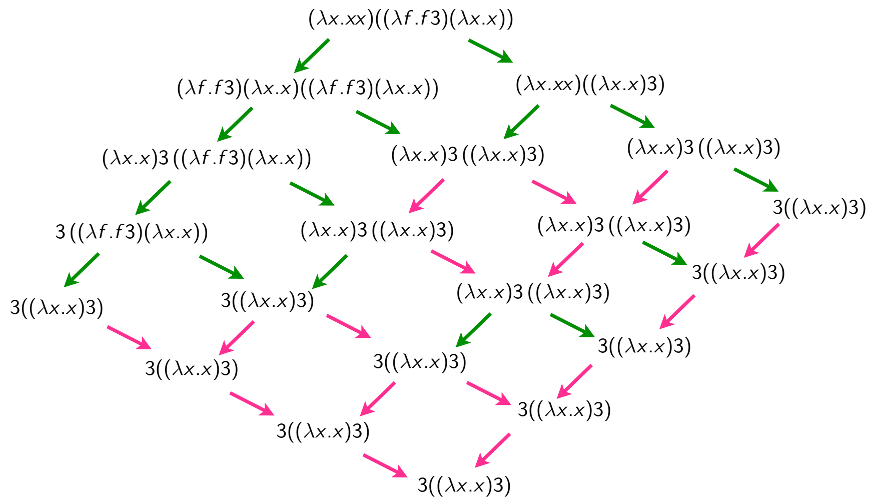


Residual of reduction (4/4)



- **Proposition** [Cube Lemma ++]:
  $\tau/(\rho \sqcup \sigma) = \tau/(\sigma \sqcup \rho)$

Equivalence by permutations (2/4)

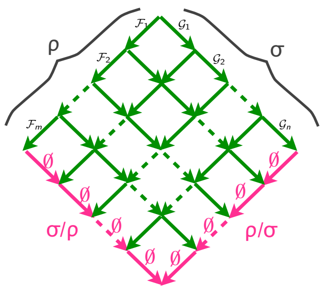


Equivalence by permutations (1/4)

- **Definition:**

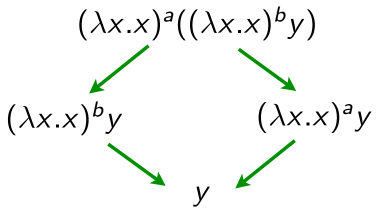
  Let ρ and σ be 2 coinitial reductions. Then ρ is equivalent to σ by permutations, $\rho \simeq \sigma$, iff:

$$\rho/\sigma = \emptyset^m \quad \text{and} \quad \sigma/\rho = \emptyset^n$$



- Notice that $\rho \simeq \sigma$ means that ρ and σ are cofinal

Equivalence by permutations (3/4)



$\rho : M = I(Iy) \xrightarrow{R_a} Iy = N$

$\sigma : M = I(Iy) \xrightarrow{R_b} Iy = N$

$\rho \not\simeq \sigma$

- Notice that $\rho \not\simeq \sigma$ while ρ and σ are coinitial and cofinal

Equivalence by permutations (4/4)

- Same with $0 \not\simeq \rho$ when $\rho : \Delta\Delta \longrightarrow \Delta\Delta$
  $\Delta = \lambda x.xx$
- **Exercice 1**: Give other examples of non-equivalent reductions between same terms
- **Exercice 2**: Show following equalities

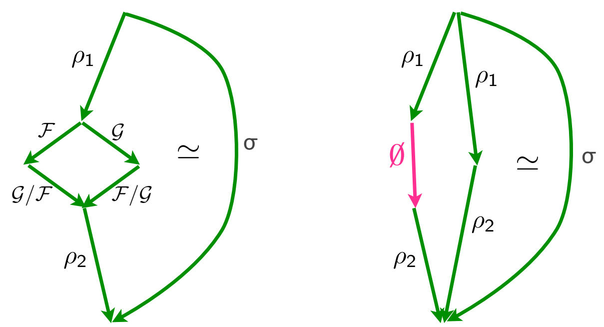
$$\rho/0 = \rho \qquad \emptyset^n/\rho = \emptyset^n$$
$$0/\rho = 0 \qquad 0 \simeq \emptyset^n$$
$$\rho/\emptyset^n = \rho \qquad \rho/\rho = \emptyset^n$$

- **Exercice 3**: Show that $\simeq$ is an equivalence relation.

Properties of equivalent reductions

- **Proposition** $\simeq$ is the smallest congruence containing

$$\mathcal{F}\,(\mathcal{G}/\mathcal{F}) \simeq \mathcal{G}\,(\mathcal{F}/\mathcal{G})$$
$$0 \simeq \emptyset$$



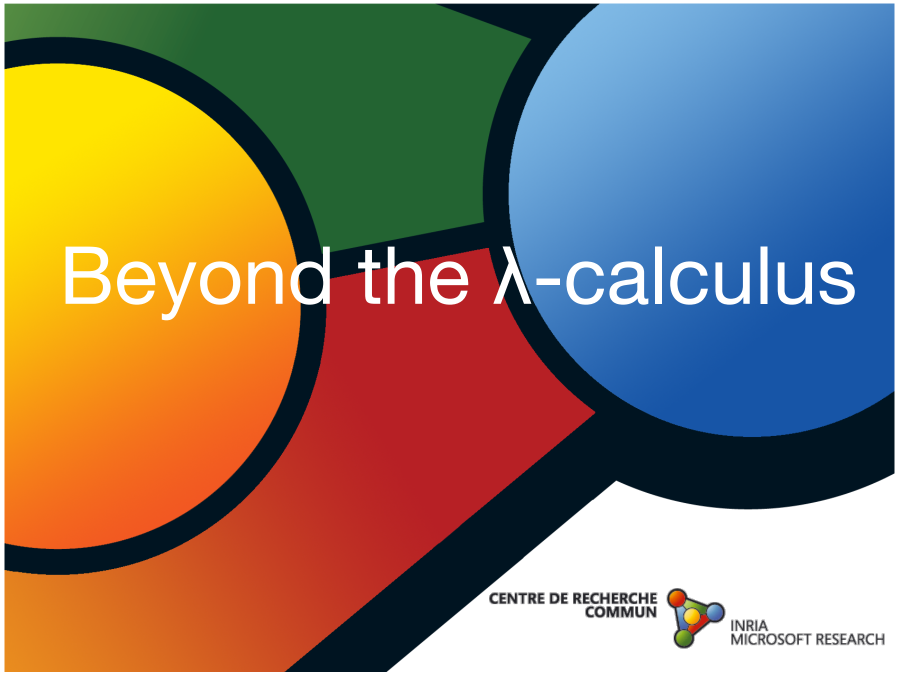
Properties of equivalent reductions

- **Proposition**

$$\rho \simeq \sigma \ \text{ iff } \ \forall \tau,\ \tau/\rho = \tau/\sigma$$
$$\rho \simeq \sigma \text{ implies } \rho/\tau \simeq \sigma/\tau$$
$$\rho \simeq \sigma \ \text{ iff } \ \tau\rho \simeq \tau\sigma$$
$$\rho \simeq \sigma \ \text{ implies } \ \rho\tau \simeq \sigma\tau$$
$$\rho \sqcup \sigma \simeq \sigma \sqcup \rho$$

- **Proof**

As $\rho \simeq \sigma$, one has $\sigma/\rho = \emptyset^n$. Therefore $\tau/\rho = (\tau/\rho)/(\sigma/\rho)$. That is $\tau/\rho = \tau/(\rho \sqcup \sigma)$. Similarly as $\sigma \simeq \rho$, one gets $\tau/\sigma = \tau/(\sigma \sqcup \rho)$. But cube lemma says $\tau/(\rho \sqcup \sigma) = \tau/(\sigma \sqcup \rho)$. Therefore $\tau/\rho = \tau/\sigma$.

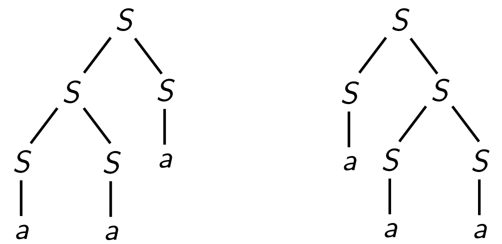


Context-free languages

- permutations of derivations in contex-free languages

$S \rightarrow SS$

$S \rightarrow a$



- each parse tree corresponds to an equivalence class

Term rewriting

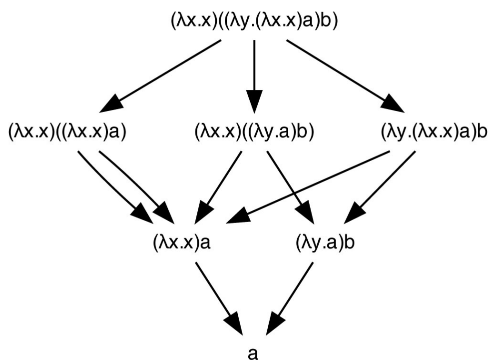
- permutations of derivations are defined with critical pairs
- critical pairs make conflicts
- only 2nd definition of equivalence works [Boudol, 1982]

Process algebras

- similar to TRS [Boudol-Castellani, 1982]

Exercices

- **Exercice 4**: Complete all proofs of propositions
- **Exercice 5**: Show equivalent reductions in



Proof Parallel moves

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Parallel moves (1/4)

- Lemma $M \xrightarrow{\mathcal{F}} N,\ P \xrightarrow{\mathcal{F}} Q \ \Rightarrow\ M\{x := P\} \xrightarrow{\mathcal{F}} N\{x := Q\}$

  Proof: exercice!

- Lemma [subst] $M\{x := N\}\{y := P\} = M\{y := P\}\{x := N\{y := P\}\}$

  when x not free in P

Parallel moves (1/4)

- Lemma $M \xrightarrow{\mathcal{F}} N,\ M \xrightarrow{\mathcal{G}} P \ \Rightarrow\ N \xrightarrow{\mathcal{F}} Q,\ P \xrightarrow{\mathcal{G}} Q$

  Proof

  Case 1: $M = x = N = P = Q$. Obvious.

  Case 2: $M = \lambda x.M_1,\ N = \lambda x.N_1,\ P = \lambda x.P_1$. Obvious by induction on M_1

  Case 3: (App-App) $M = M_1 M_2,\ N = N_1 N_2,\ P = P_1 P_2$. Obvious by induction on $M_1,\ M_2$.

  Case 4: (Red'-Red') $M = (\lambda x.M_1)^a M_2,\ N = (\lambda x.N_1)^a N_2,\ P = (\lambda x.P_1)^a P_2,\ a \notin \mathcal{F} \cup \mathcal{G}$

  Then induction on $M_1,\ M_2$.

  Case 4: (beta-Red') $M = (\lambda x.M_1)^a M_2,\ N = N_1\{x := N_2\},\ P = (\lambda x.P_1)^a P_2,\ a \in \mathcal{F}, a \notin \mathcal{G}$

  By induction $N_1 \xrightarrow{\mathcal{G}} Q_1,\ P_1 \xrightarrow{\mathcal{F}} Q_1$. And $N_2 \xrightarrow{\mathcal{G}} Q_2,\ P_1 \xrightarrow{\mathcal{F}} Q_2$.

  By lemma, $N_1\{x := N_2\} \xrightarrow{\mathcal{G}} Q_1\{x := Q_2\}$. And $(\lambda x.P_1)^a P_2 \xrightarrow{\mathcal{F}} Q_1\{x := Q_2\}$

  Case 5: (beta-beta) $M = (\lambda x.M_1)^a M_2,\ N = N_1\{x := N_2\},\ P = P_1\{x := P_2\},\ a \in \mathcal{F} \cap \mathcal{G}$

  As before with same lemma.