On the logical structure of choice and bar induction principles

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joint work with Nuria Brede

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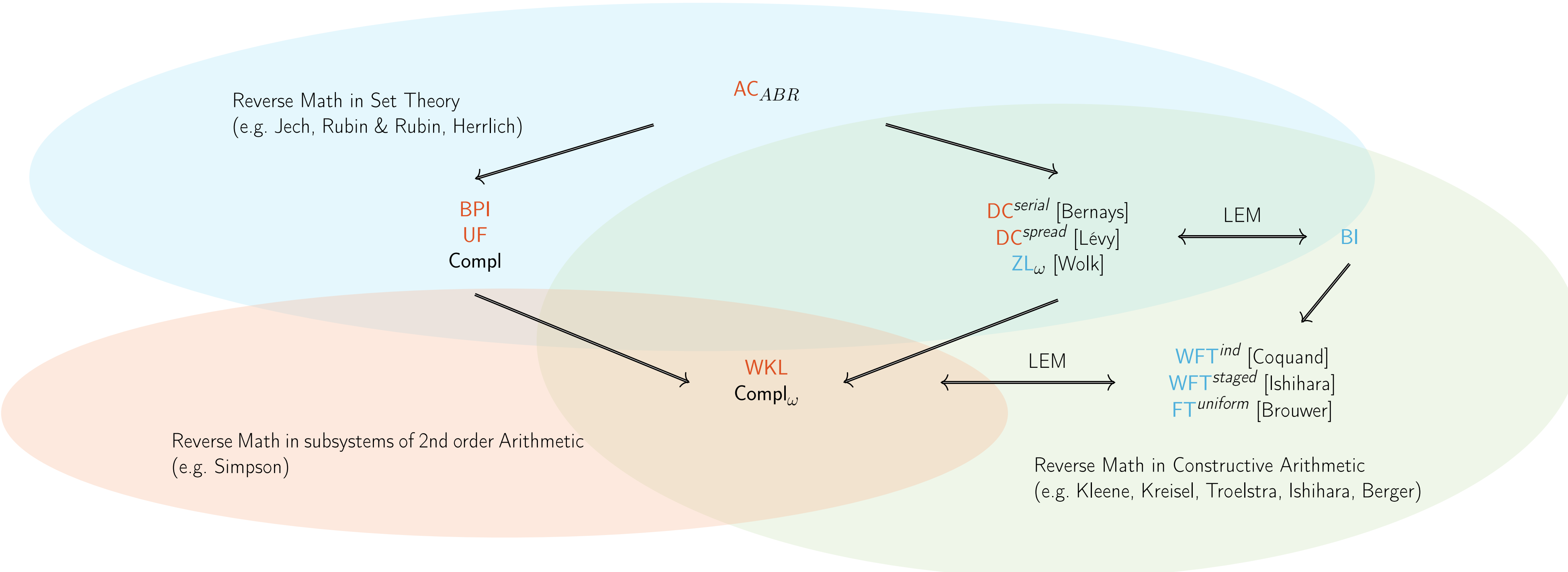
(includes post-seminar errata)

Talk based on the paper *On the logical structure of choice and bar induction principles, LICS'21*, with a few refinements

Standard results about the axiom of choice



BPI	=	Boolean Prime Ideal Theorem
UF	=	Ultrafilter Theorem
AC	=	Axiom of Choice
DC	=	Axiom of Dependent Choice
WKL	=	Weak Kőnig's Lemma

ZL_ω	=	Countable Zorn's Lemma
BI	=	Bar Induction
(W)FT	=	(Weak) Fan Theorem
Compl	=	Gödel's Completeness Theorem

Use **logical duality** as guiding classification principle:

choice principles	bar induction principles
ill-foundedness properties	barredness properties

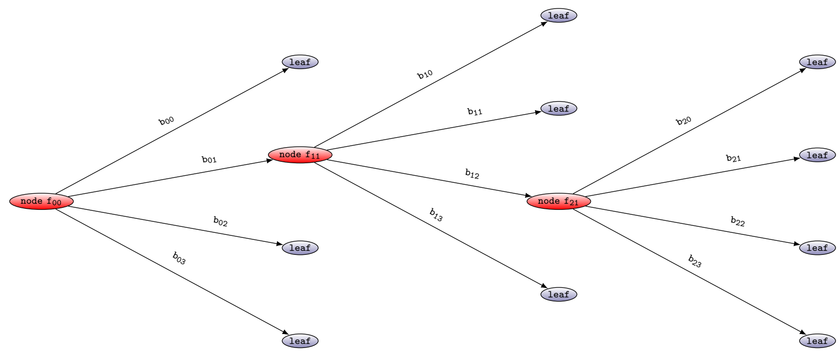
considered as **extensionality schemes**

effective $\Rightarrow$ observational	observational $\Rightarrow$ effective

Different definitions of well-founded tree

An intrinsically well-founded definition of tree

A simple **"effective"** definition: well-founded tree as an **inductive type**

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Inductive wftree :=
| leaf : wftree
| node : (B → wftree) → wftree
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Trees (and their negative) as predicates

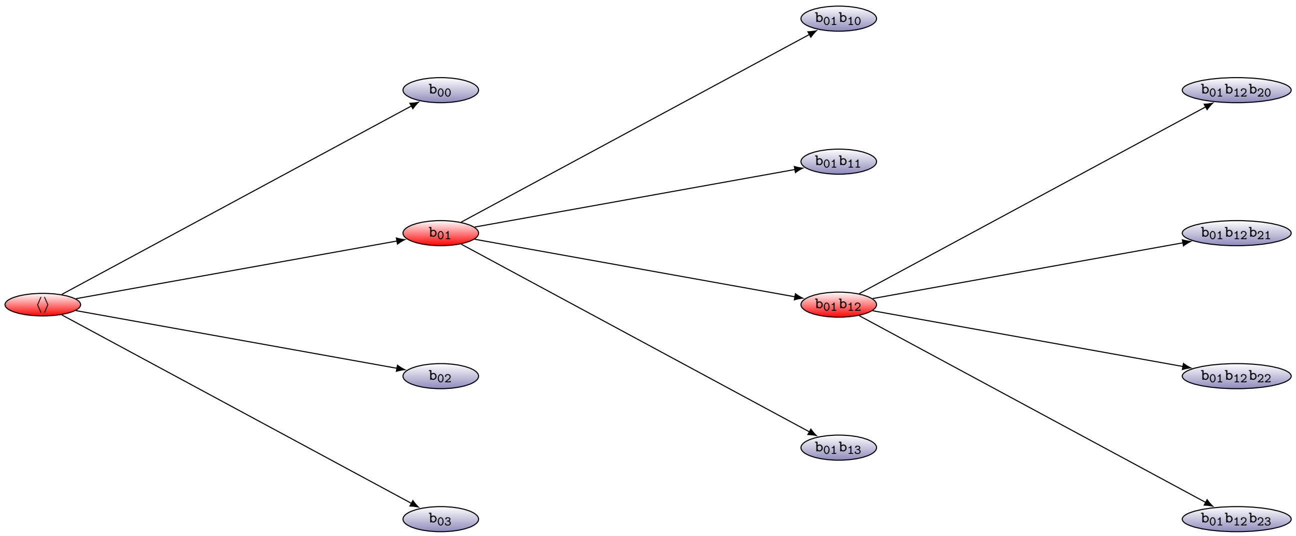
Let B be a domain and u ranges over the set B^* of finite sequences of elements of B. We write $\langle\rangle$ for the empty sequence and $u \star b$ for the extension with one element. We define:

T is a tree	T is monotone
(closure under restriction)	(closure under extension)
$\forall u \, \forall a \, (u \star a \in T \Rightarrow u \in T)$	$\forall u \, \forall a \, (u \in T \Rightarrow u \star a \in T)$

From trees as inductive types to trees as predicates

To any inductively-defined tree t, we can associate a tree-as-predicate $t^{\#}$ by recursion on t as follows:

$$
\begin{array}{lcl}
u \in \mathtt{leaf}^{\#} & \triangleq & \bot \\
u \in \mathtt{node}(f)^{\#} & \triangleq & u = \langle\rangle \vee \exists a\, \exists u'\, (u = a@u' \wedge u' \in f(a)^{\#})
\end{array}
$$



Two characterisations of a well-founded tree-as-predicate

Effective characterisation of a well-founded tree-as-predicate

T *has an inductive skeleton*

$$\exists t : \texttt{wftree}\,(T = t^{\#})$$

Effective characterisation of a well-founded tree-as-predicate

T *has an inductive skeleton*

$$\exists t : \texttt{wftree}\,(T = t^{\#})$$

which can be equivalently bundled into

T *inductively well-founded* is short for inductively well-founded at $\langle\rangle \in A^*$

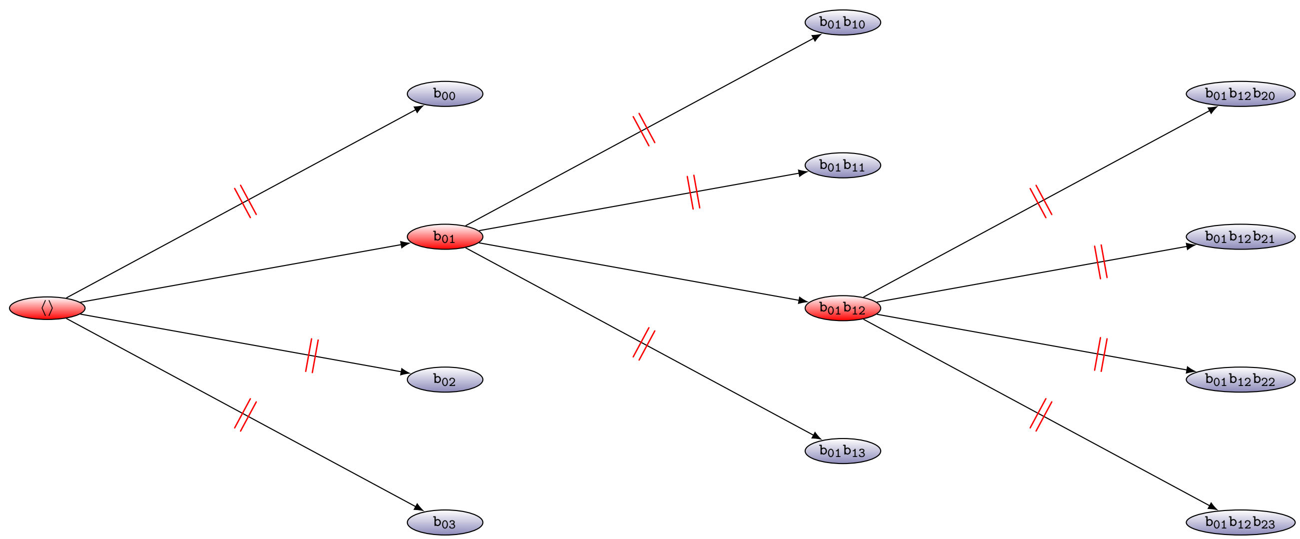
T *inductively well-founded at* u holds when:

- $u \notin T$
- *or*, recursively, for all a, T is *inductively well-founded at* $u \star a$

Observational characterisation of a well-founded tree-as-predicate

T *observationally well-founded*

$$\forall \beta \in \mathbb{N} \to B.\ \exists n \in \mathbb{N}.\ \neg T(\beta_{|n})$$



Two characterisations of a well-founded tree-as-predicate

- From the **"effective"** representation of a well-founded tree we can always construct a predicate that is an **"observational"** representation of the tree
- To conversely obtain an effective representation of a tree T from its observational representation requires an axiom:

$$T \textit{ observationally well-founded} \implies T \textit{ inductively well-founded}$$

Bar Induction

If instead we build the *negative* of a tree-as-predicate and restate well-foundedness on the negative tree, one obtain bar induction:

- T *inductively well-founded* is the same as $\neg T$ *inductively barred*
- T *observationally well-founded* is the same $\neg T$ *barred*

- Bar Induction says that for a type B and a tree T,

$$\underbrace{T \text{ barred}}_{observational} \implies \underbrace{T \text{ inductively barred}}_{effective}$$

Dually: ill-foundedness

Dually, ill-foundedness of a tree T can be defined in different ways.

Let us concentrate on the finite-branching case. We have:

Effective view

$$T \text{ is staged infinite} \quad \triangleq \quad \forall n\, \exists u\, |u| = n \wedge u \in T$$

Observational view

$$T \text{ has an infinite branch} \quad \triangleq \quad \exists \alpha\, \forall u \leq \alpha\, T(u)$$

Weak Kőnig's Lemma connects the two views (when B is $\mathbb{B}\mathsf{ool}$):

$$\mathsf{WKL}_T \quad \triangleq \quad T \text{ is staged infinite} \Rightarrow T \text{ has an infinite branch}$$

Observation: a diversity of definitions for the "effective" versions of "barred"/"well-founded" and "ill-founded"

Kőnig's Lemma:	T is staged infinite $\Rightarrow$ T has an infinite branch	(B finite)
C_{WKL}:	T is a spread $\Rightarrow$ T has an infinite branch	(J. Berger, $B = \mathbb{B}\text{ool}$)

Fan Theorem:	T barred $\Rightarrow$ T uniformly barred	(B finite, Brouwer)
Fan Theorem:	T barred $\Rightarrow$ T staged barred	(B finite, Ishihara)

- *having an infinite branch* is the exact dual to *barred*
- the dual of *inductively barred* is equivalent to the existence of a *spread* subset
- *being staged infinite* is dual to *uniformly barred* up to asking for T to be a tree
- *uniformly barred* and *having unbounded paths* are respectively intuitionistically and cointuitionistically equivalent to *inductively barred* and its dual productive for finitely-branching trees

Giving a name to these definitions

T is progressing[1] at u $u \in T \Rightarrow (\exists a\, u \star a \in T)$	T is hereditary at u $(\forall a\, u \star a \in T) \Rightarrow u \in T$
T is progressing[1] $\forall u\, (T$ is progressing at $u)$	T is hereditary $\forall u\, (T$ is hereditary at $u)$

	Dual concepts on dual predicates		
	ill-foundedness	*barredness-style*	
	Effective concepts (finite-branching only)		
	T has unbounded paths $\forall n\, \exists u\, (\|u\| = n \wedge \forall v\, (v \leq u \Rightarrow v \in T))$	T is uniformly barred $\exists n\, \forall u\, (\|u\| = n \Rightarrow \exists v\, (v \leq u \wedge v \in T))$	used in Fan Theorem
used in Kőnig's Lemma	T is staged infinite[1] $\forall n\, \exists u\, (\|u\| = n \wedge u \in T)$	T is staged barred[1] $\exists n\, \forall u\, (\|u\| = n \Rightarrow u \in T)$	alt. used in Fan Theorem
	Effective concepts (arbitrary branching)		
used in C_{WKL}	T is a spread $\langle\rangle \in T \wedge T$ progressing	T is barricaded[1] T hereditary $\Rightarrow \langle\rangle \in T$	
	T is productive $\langle\rangle \in \nu X.\lambda u.(u \in T \wedge \exists b\, u \star b \in X)$	T is inductively barred $\langle\rangle \in \mu X.\lambda u.(u \in T \vee \forall b\, u \star b \in X)$	used in Bar Induction
	Observational concepts		
	T has an infinite branch $\exists \alpha\, \forall u\, (u$ initial segment of $\alpha \Rightarrow u \in T)$	T is barred $\forall \alpha\, \exists u\, (u$ initial segment of $\alpha \wedge u \in T)$	

[1]Not being aware of an established terminology for this concept, we use here our own terminology.

Giving the central rôle to *inductively barred* and its dual

We focus on the definition of the dual of inductively barred and on its dual productive. In full:

T is productive (short for productive from $\langle\rangle \in B^*$)

T productive from $u \in B^*$ holds when:

- u is in T
- *and*, recursively, there is $b \in B$ such that T productive from $u \star b$

Giving the central rôle to *inductively barred* and its dual

Bar induction (BI_{BT})

T barred $\Rightarrow$ T inductively barred

Tree-Based Dependent Choice (DC_{BT}^{prod})

T productive $\Rightarrow$ T has an infinite branch

Recovering standard principles

$\mathsf{WKL}_T \iff \mathsf{DC}^{prod}_{\mathbb{B}\mathsf{ool}T}$ up to classical (actually co-intuitionistic) reasoning

$\mathsf{WFT}_T \iff \mathsf{BI}_{\mathbb{B}\mathsf{ool}T}$ up to intuitionistic reasoning

$\mathsf{DC}^{serial}_{BRb_0} \iff \mathsf{DC}^{prod}_{BR^{\triangleright}(b_0)}$

where

$$u \in R^{\triangleright}(b_0) \triangleq \text{case } u \text{ of } \begin{bmatrix} \langle\rangle & \mapsto & \top \\ b & \mapsto & R(b_0, b) \\ u' \star b \star b' & \mapsto & R(b, b') \end{bmatrix}$$

$$\mathsf{DC}^{serial}_{BRb_0} \quad \triangleq \forall b \exists b' \, R(b, b') \Rightarrow \exists \alpha \, (\alpha(0) = b_0 \wedge \forall n \; R(\alpha(n), \alpha(n+1)))$$

Relaxing the sequentiality

Let A and B be domains. Let now use v to range over the set $(A \times B)^*$ of finite sequences of pairs of elements in A and B.

We say $(a, b) \in v$ if (a, b) is one of the components of v.

We write $v \leq v'$ is v is included in v' when seen as sets.

For $v \in (A \times B)^*$, we write $dom(v)$ for the set of a such that there is some b such that $(a, b) \in v$.

If $\alpha \in A \to B$, we write $v \prec \alpha$ and say that v is a finite approximation of α if $\alpha(a) = b$ for all $(a, b) \in v$.

Let T be a predicate on $(A \times B)^*$. We write $\downarrow T$ and $\uparrow T$ to mean the following inner and outer closures with respect to $\leq$:

$$v \in \downarrow T \triangleq \forall v' \leq v \; (v' \in T)$$

$$v \in \uparrow T \triangleq \exists v' \leq v \; (v' \in T)$$

Relaxing the sequentiality (effective view)

T inductively A-B-barred from $v \in (A \times B)^*$ holds when:

- v is in the outer closure of T
- *or*, recursively, there exists $a \notin dom(v)$ such that for all $b \in B$, T is inductively A-B-barred from $v \star (a, b)$

T coinductively A-B-approximable from $v \in (A \times B)^*$ holds when:

- v is in the inner closure of T
- *and*, recursively, for all $a \notin dom(v)$, there is $b \in B$ such that T is coinductively A-B-approximable from $v \star (a, b)$

Relaxing the sequentiality (observational view)

T A-B-barred if $\forall \alpha \in A \to B \; \exists v \prec \alpha \; (v \in T)$

T has an A-B-choice function if $\exists \alpha \in A \to B \; \forall v \prec \alpha \; (v \in T)$

This leads to the following generalisation

Generalised Bar Induction (GBI_{ABT})

$$\underbrace{T\ A\text{-}B\text{-}\textit{barred}}_{observational} \implies \underbrace{T\ A\text{-}B\text{-}\textit{inductively barred}}_{effective}$$

Generalised Dependent Choice (GDC_{ABT})

$$\underbrace{T\ \textit{coinductively}\ A\text{-}B\text{-}\textit{approximable}}_{effective} \implies \underbrace{T\ \textit{has an}\ A\text{-}B\text{-}\textit{choice function}}_{observational}$$

Results justifying the generalisation

$$\mathsf{GBI}_{\mathbb{N}BT} \iff \mathsf{BI}_{BT}$$

$$\mathsf{GDC}_{\mathbb{N}BT} \iff \mathsf{DC}_{BT}^{prod}$$

Actually, GBI_{ABT} and GDC_{ABT} could be further generalised into schemes $\mathsf{GBI}_{ABT\leq}$ and $\mathsf{GDC}_{ABT\leq}$ such that instantiating the order with the prefix order on approximations of $\mathbb{N} \to B$ gives BI_{BT} and DC_{BT}^{prod} while instantiating the order with the inclusion order gives GBI_{ABT} and GDC_{ABT}.

The Boolean Prime Ideal Theorem

The specialisation to $\mathbb{B}$**ool** of the generalisation also captures the Boolean Prime Ideal Theorem.

Let $(\mathcal{B}, \vee, \wedge, \bot, \top, \neg, \vdash)$ be a Boolean algebra and I an ideal on $\mathcal{B}$. We extend I on $(\mathcal{B} \times \mathbb{B}\mathsf{ool})^*$ by setting $u \in I^+$ if $(\bigvee_{(b,0)\in u} \neg b) \vee (\bigvee_{(b,1)\in u} b) \in I$. We have:

$$\mathsf{GDC}_{\mathcal{B}\mathbb{B}\mathsf{ool}I^+} \iff \mathsf{BPI}_{\mathcal{B},I}$$

The full axiom of choice

Let AC_{ABR} be $\forall a^A \exists b^B R(a,b) \Rightarrow \exists \alpha^{A \to B} \forall a^A R(a, \alpha(a))$

Define the *positive alignment* $R_\top$ of R by

$$R_\top \triangleq \lambda u. \forall (a,b) \in u\, R(a,b)$$

Then, AC_{ABR} arrives as the instance $\mathsf{GDC}_{ABR_\top}$

Strength of the generalisation

Without further restrictions, GDC and GBI are inconsistent:

- Take $A \triangleq \mathbb{N} \rightarrow \mathbb{B}\text{ool}$
- Take $B \triangleq \mathbb{N}$
- Define T so that it constrains a choice function to be injective:

$$v \in T \triangleq \forall f f' n, ((f, n) \in v) \wedge ((f', n) \in v) \Rightarrow f = f'$$

Then, in the case of GDC, a coinductive A-B-approximation can always be found but an A-B-choice function would be an injective function from $\mathbb{N} \rightarrow \mathbb{B}\text{ool}$ to $\mathbb{N}$, what is inconsistent.

A consistent restriction

A naive restriction is to require that:

- either A is countable
- or B is finite
- or T is atomic (or unary), meaning for all u and v:
  - in the ill-founded case $u \in T \wedge v \in T \Rightarrow u \cup v \in T$
  - in the barred case $u \cup v \in T \Rightarrow u \in T \vee v \in T$

The restriction preserves the previous instantiations and makes GDC equivalent to AC since it implies AC, and, conversely, each of its three restrictions is implied by a consequence of AC.

Dually for GBI.

Summary of main results

$\mathsf{AC}_{ABR} \triangleq \mathsf{GDC}_{ABR_\top}$

$\mathsf{BPI}_{AI} \triangleq \mathsf{GDC}_{A\mathbb{B}\mathsf{ool}\mathcal{T}_I}$

$\mathsf{Compl}^-_{A\mathcal{T}} \triangleq \mathsf{GDC}_{A\mathbb{B}\mathsf{ool}\mathcal{T}^C}$

$\mathsf{Compl}^+_{A\mathcal{T}} \triangleq \mathsf{GBI}_{A\mathbb{B}\mathsf{ool}\mathcal{T}^C}$

$\mathsf{DC}^{serial}_{BRb} \triangleq \mathsf{GDC}_{\mathbb{N}BR^{\triangleright}_\top}$

$\mathsf{DC}^{prod}_{BT} \triangleq \mathsf{GDC}_{\mathbb{N}BT}$

$\mathsf{ZL}_{\mathbb{N}R} \triangleq \mathsf{GBI}_{\mathbb{N}B(\neg R)^*_\bot}$

$\mathsf{BI}_{BT} \triangleq \mathsf{GBI}_{\mathbb{N}BT}$

$\mathsf{WKL}^{prod}_{BT} \triangleq \mathsf{GDC}_{\mathbb{N}\mathbb{B}\mathsf{ool}T}$

$\mathsf{WFT}^{ind}_{BT} \triangleq \mathsf{GBI}_{\mathbb{N}\mathbb{B}\mathsf{ool}T}$

AC	=	Axiom of Choice
DC	=	Axiom of Dependent Choice
BPI	=	Boolean Prime Ideal Theorem
Compl$^-$	=	Completeness (consistent ⇒ model)
WKL	=	Weak Kőnig's Lemma

Compl$^+$	=	Completeness (valid ⇒ provable)
ZL	=	Zorn's Lemma
BI	=	Bar Induction
WFT	=	Weak Fan Theorem

Remarks and perspectives

Studying the principles together with their dual allow to see where non-linear reasoning is used. For instance, that the equivalence between WKL_T^{staged} and $\mathsf{GDC}_{\mathbb{NB}\mathsf{ool}T}$ is essentially classical means that the equivalence between $\mathsf{WFT}^{uniform}$ and $\mathsf{GBI}_{\mathbb{NB}\mathsf{ool}T}$ is essentially non-linear. And conversely, that the latter is intuitionistic says that the former only requires the co-intuitionistic reasoning part of classical logic.

Other variants of choice can probably be added to the picture:

- U. Berger's update induction on functions in $\mathbb{N} \to B$ for open predicates seems to directly generalize to updates of functions on $A \to B$ for predicates of finite character (i.e. of the form $\forall v \prec \alpha\,(v \in T)$ or $\exists v \prec \alpha\,(v \in T)$), giving a well-founded induction principle, or dually, maximal approximations.
- generalisations of hybrid forms such as J. Berger's C_{Fan} seem also to be rather canonical:

$$T \text{ coinductively approximable} \wedge U \text{ barred} \Rightarrow \exists u\,(u \in T \wedge u \in U)$$
$$T \text{ has a choice function} \wedge U \text{ inductively barred} \Rightarrow \exists u\,(u \in T \wedge u \in U)$$