A unified logical structure to the axiom of choice, bar induction and some of their relatives

Hugo Herbelin

joint work with Nuria Brede

Proof, Computation, Complexity

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Outline

- Bar induction as an extensional principle

- A systematic study of bar induction and dependent choice principles as dual principles

- A common generalisation to the Ultrafilter Theorem and Dependent Choice

Bar Induction as an extensional principle

Notations

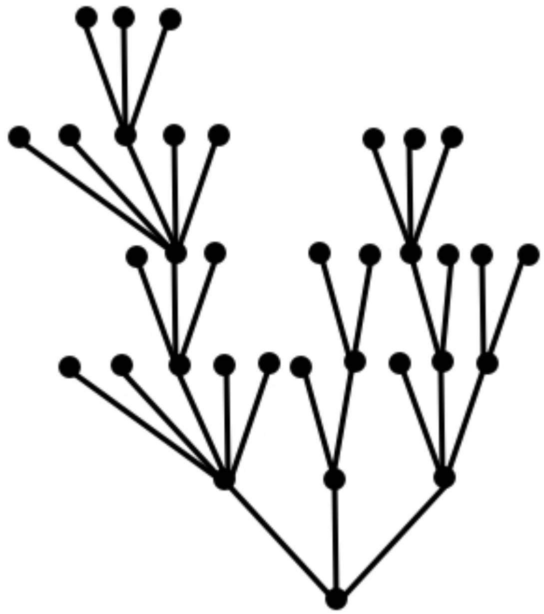
Let B be a domain and u ranges over the set B^* of finite sequences of elements of B. We write $\langle\rangle$ for the empty sequence, $u \star b$ for the extension with one element, $|u|$ for the length of u, **nth** $p\ u$ for the p-th element of u counting from 0 when $|u| > p$ and $u@u'$ for the concatenation.

A predicate T over B^* is a *tree* if it is closed by restriction (i.e. if $u \star b \in T$ implies $u \in T$).

A predicate U over B^* is *monotone* if it is closed by extension (i.e. if $u \in U$ implies $u \star b \in U$).

We may alternatively use the notation $T(u)$ for $u \in T$.

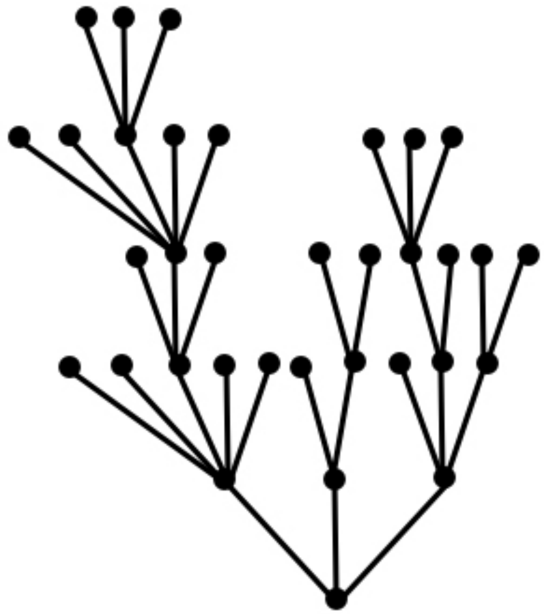
Bar Induction



This is a well-founded tree

(characterised by: T well-founded $\triangleq \forall\alpha\,\exists n\,\neg T(\alpha_{|n})$)

Bar Induction



This is also a well-founded tree

(but characterised as some **wftree** "realising" T where **wftree** smallest set generated from

$$\begin{aligned} &\textsf{leaf} : \textsf{wftree} \\ &\textsf{node} : (B \to \textsf{wftree}) \to \textsf{wftree} \end{aligned}$$

)

From **wftree** to a well-founded tree-as-predicate

From a **wftree** t, a predicate $T_{\textsf{tree}}(t)$ representing t as a predicate can be defined by induction on t:

$$T_{\textsf{tree}}(t) \triangleq \lambda u.T'_{\textsf{tree}}(t, u)$$

where

$$\begin{array}{lcl} T'_{\textsf{tree}}(\textsf{leaf}, u) & \triangleq & \bot \\ T'_{\textsf{tree}}(\textsf{node}(f), \langle\rangle) & \triangleq & \top \\ T'_{\textsf{tree}}(\textsf{node}(f), b \mathbin{@} u') & \triangleq & T'_{\textsf{tree}}(f(b), u') \end{array}$$

The proof that it is well-founded is:

$$\begin{array}{lcl} wf(t) & : & \forall\alpha\,\exists n\,\neg T'_{\textsf{tree}}(t, \alpha_{|n}) \\ wf(t) & \triangleq & \lambda\alpha.\, wf'(t, \alpha) \end{array}$$

where

$$\begin{array}{lcl} wf'(\textsf{leaf}, \alpha) & \triangleq & (0, \lambda a.a) \\ wf'(\textsf{node}(f), \alpha) & \triangleq & \mathbf{let}\ (n, p) = wf'(f(\alpha(0)), \lambda n.\alpha(n+1))\ \mathbf{in}\ (n+1, p) \end{array}$$

Bar Induction seen as a converse of the translation from **wftree** to a well-founded tree-as-predicate

Bar Induction can be seen as stating the existence of a converse translation:

$$T \text{ well-founded } \rightarrow \exists t\, \forall v\, (T(v) \rightarrow T'_{\mathsf{tree}}(t, v))$$

It can be realised by a recursive loop, namely bar recursion:

For $H : \forall \alpha\, \exists n\, \neg T(\alpha_{|n})$ and $b_0 \in B$ and $istree : \forall v w\, (T(v@w) \rightarrow T(v))$, we set:

$$\begin{array}{lcl} barrec'(H, u) & : & \exists t\, \forall v\, (T(u\,@\,v) \rightarrow T'_{\mathsf{tree}}(t, u\,@\,v)) \\ barrec'(H, u) & \triangleq & \text{if } \pi_1(H\ \widehat{u}) = |u| \text{ then } (\mathsf{leaf}, H_1) \text{ else } (\mathsf{node}\ (\lambda b. barrec'(H, u \star b)), H_2) \\ & & \\ barrec(H) & : & \exists t\, \forall v\, (T(v) \rightarrow T'_{\mathsf{tree}}(t, v)) \\ barrec(H) & \triangleq & barrec'(H, \langle\rangle) \end{array}$$

where $\widehat{u} \triangleq \lambda p.\text{if } p < |u| \text{ then } \mathsf{nth}\ p\ u \text{ else } b_0$ and H_1 and H_2 are well-chosen proofs.

Bar Induction seen as a well-foundedness theorem

Let $\overline{T}$ be the *inductive downwards closure* of the complement of T. We can define it as the least fixed point $\overline{T} \triangleq \mu X.\lambda u.(u \notin T \vee \forall b\,(u \star b \in X))$, or, equivalently, using inference rules:

$$\frac{u \notin T}{u \in \overline{T}} \qquad \frac{\forall b \in B\ u \star b \in \overline{T}}{u \in \overline{T}}$$

We say that T is *inductively well-founded* if $\langle\rangle \in \overline{T}$.

Statement of Bar Induction seen as an extensionality property of well-foundedness:

$$BI_{BT}^{\mathrm{wf}} \quad \triangleq \quad T \text{ well-founded} \rightarrow T \text{ inductively well-founded}$$

Bar Induction (original version)

Original bar induction is the statement about the complement of a tree.

Let $\overline{U}$ be now the *inductive downwards closure* of some U thought as the complement of a tree:

$$\frac{u \in U}{u \in \overline{U}} \qquad \frac{\forall b \in B\; u \star b \in \overline{U}}{u \in \overline{U}}$$

We say that U is *inductively barred* if $\langle\rangle \in \overline{U}$.

We say that U is *barred* if $\forall \alpha\, \exists n\, U(\alpha_{|n})$

Statement of Bar Induction:

$$BI_{BU} \quad \triangleq \quad U \text{ barred} \rightarrow U \text{ inductively barred}$$

A systematic study of bar induction and dependent choice principles as dual principles

Fan Theorem: a standard consequence of Bar Induction

Fan Theorem can be proved to be a specialisation of Bar Induction to the finitely-branching case. The standard way to state Fan Theorem is however as:

FT_{BU}: U barred $\rightarrow$ U uniformly barred (B finite)

where, uniformly barred is defined by:

$$\exists n \, \forall u \, (|u| = n \rightarrow \exists v \, (v \leq u \wedge v \in U))$$

The equivalence between U uniformly barred and U inductively barred is easy to prove intuitionistically (for B finite).

Note: There are different formulations of Fan Theorem. For instance, another formulation allows B to be a different finite domain at each step. Also, the different ways to intuitionistically define "finite" lead to statements of different strengths (see Veldman).

Weak Fan Theorem: another standard consequence of Bar Induction

Weak Fan Theorem is the specialisation of Fan Theorem to the Boolean case:

$WFT_U = FT_{\mathbb{B}oolU}$: U barred $\rightarrow$ U uniformly barred

Weak Fan Theorem: another standard consequence of Bar Induction

Note that there are three possible representation of an infinite function from $\mathbb{N}$ to $\mathbb{B}ool$ which leads to three formulations of WFT of different strengths:

- It can be represented as a function, i.e. as a functional object of type $\mathbb{N} \to \mathbb{B}ool$, leading to WFT^{fun}.
- It can be represented as a functional relation, i.e. as a relation R over $\mathbb{N} \times \mathbb{B}ool$ such that $\forall n\, \exists! a\, R(n,a)$, leading to $WFT^{\mathsf{fun-rel}}$.
- It can be represented as predicate P over $\mathbb{N}$ with intended meaning 0 when $P(n)$ holds and 1 when $\neg P(n)$ holds (and unknown meaning otherwise), leading to WFT^{pred}.

We have:

$$WFT^{\mathsf{fun}} \quad \underset{\text{(but converse needs AC!)}}{\Rightarrow} \quad WFT^{\mathsf{fun-rel}} \quad \underset{\text{(but converse needs LEM)}}{\Rightarrow} \quad WFT^{\mathsf{pred}}$$

In particular, WFT^{pred} is *provable* in intuitionistic second-order arithmetic.

Dually: ill-foundedness

Dually, ill-foundedness of a tree T can be defined in different ways. Let us concentrate on Weak König's Lemma which is the dual of the Weak Fan Theorem for T a tree:

Intensional view

$$T \text{ is staged infinite} \quad \triangleq \quad \forall n \, \exists u \, (|u| = n \wedge u \in T)$$

which is the dual of uniformly barred

Extensional view

$$T \text{ has an infinite branch} \quad \triangleq \quad \exists \alpha \, \forall n \, T(\alpha_{|n})$$

which is the dual of barred

Weak König's Lemma connects the two views (when B is $\mathbb{B}ool$):

$$WKL_T \quad \triangleq \quad T \text{ is staged infinite} \rightarrow T \text{ has an infinite branch}$$

As for WFT, one can define three variants WKL^{fun}, $WKL^{\mathsf{fun-rel}}$ and WKL^{pred}. The latter is provable in intuitionistic second-order arithmetic for decidable T!

Other definitions for the "intensional" versions of "well-founded" and "ill-founded"

In Berger (2009), we find the following variant of WKL, called C_{WKL}, which can also be thought as a form of dependent choices on $\mathbb{B}$**ool**, thus motivating Ishihara, Berger, Schuster (2005, 2012) to call it $\mathrm{DC}^{\vee}$:

$$C_{WKL}/\mathrm{DC}^{\vee}\text{:}\ \ T \text{ is a spread} \to T \text{ has an infinite branch}$$

where

$$T \text{ is a spread } \triangleq \langle\rangle \in T \wedge \forall u\,(u \in T \to \exists a\, u \star a \in T)$$

We can then show that the dual of *inductively barred* for a predicate is equivalent to the existence of a *spread* subset of the dual predicate.

In passing, we can also show that *uniformly barred* and *having arbitrary long branches* are respectively intuitionistically and cointuitionistically equivalent to *inductively barred* and its dual for finitely-branching trees.

Giving a name to these definitions

Equivalent concepts on dual predicates	
T is a tree $\forall u\, \forall a\, (u \star a \in T \to u \in T)$	U is monotone $\forall u\, \forall a\, (u \in U \to u \star a \in U)$
T is progressing $\forall u\, (u \in T \to \exists a\, u \star a \in T)$	U is hereditary $\forall u\, ((\forall a\, u \star a \in U) \to u \in U)$

	Dual concepts on dual predicates		
	infinite-branch-style (ill-foundedness)	*well-foundedness-style*	
	Intensional concepts (finite-branching only)		
	T has arbitrary long branches $\forall n\, \exists u\, (\lvert u\rvert = n \wedge \forall v\, (v \leq u \to v \in T))$	U is uniformly barred $\exists n\, \forall u\, (\lvert u\rvert = n \to \exists v\, (v \leq u \wedge v \in U))$	used in Fan Theorem
used in König's Lemma	T is staged infinite[1] $\forall n\, \exists u\, (\lvert u\rvert = n \wedge u \in T)$	U is staged barred[1] $\exists n\, \forall u\, (\lvert u\rvert = n \to u \in U)$	
	Intensional concepts (arbitrary branching)		
used in C_{WKL}	T is a spread $\langle\rangle \in T \wedge T$ progressing	U is resisting[1] U hereditary $\to \langle\rangle \in U$	
	T is coinductively choosable $\langle\rangle \in \nu X.\lambda u.(u \in T \wedge \exists b\, u \star b \in X)$	U is inductively barred $\langle\rangle \in \mu X.\lambda u.(u \in U \vee \forall b\, u \star b \in X)$	used in Bar Induction
	Extensional concepts		
	T has an infinite branch $\exists \alpha\, \forall u\, (u \text{ initial segment of } \alpha \to u \in T)$	U is barred $\forall \alpha\, \exists u\, (u \text{ initial segment of } \alpha \wedge u \in U)$	

[1] Not being aware of an established terminology for this concept, we use here our own terminology.

From now on, giving the central rôle to *inductively barred* and its dual

We focus on the definition of the dual of *inductively barred*. Let $\underline{T}$ be the coinductively-defined *pruning* (or *interior*) of T:

$$\frac{u \in T \qquad \exists b \in B\, u \star b \in \underline{T}}{u \in \underline{T}}$$

Then, T is *coinductively choosable* $\quad \triangleq \quad \langle\rangle \in \underline{T}$.

Giving the central rôle to *inductively barred* and its dual

Bar induction (BI_{BU})

U barred $\rightarrow$ U inductively barred

Tree-Based Dependent Choice (TDC_{BT})

T coinductively choosable $\rightarrow$ T has an infinite branch

Recovering standard principles

$WKL_T \iff TDC_{\mathbb{B}\mathsf{ool}T}$ up to classical (actually co-intuitionistic) reasoning

$WFT_U \iff BI_{\mathbb{B}\mathsf{ool}U}$ up to intuitionistic reasoning

$DC_{BRb_0} \iff TDC_{BR^*(b_0)}$

where

$$u \in R^*(b_0) \triangleq \text{case } u \text{ of } \begin{bmatrix} \langle\rangle & \mapsto & \top \\ b & \mapsto & R(b_0, b) \\ u' \star b \star b' & \mapsto & R(b, b') \end{bmatrix}$$

$$DC_{BRb_0} \quad \triangleq \forall b \exists b' \, R(b, b') \to \exists \alpha \, (\alpha(0) = b_0 \land \forall n \; R(\alpha(n), \alpha(n+1)))$$

A common generalisation to the Ultrafilter Theorem and Dependent Choice

Standard results about the axiom of choice

(classical set theory)

Axiom of Choice (AC_{ABR})

$$\forall a^A \, \exists b^B \, R(a,b) \rightarrow \exists \alpha^{A \rightarrow B} \, \forall a^A \, R(a, \alpha(a))$$

$\swarrow$ $\qquad\qquad$ $\searrow$

Ultrafilter Theorem ($UF_{\mathcal{B}F}$)
$=$
Gödel's completeness
(on arbitrary large languages)

Axiom of Dependent Choices (DC_{BR})
$\downarrow$
Axiom of Countable Choice ($AC_{\mathbb{N}BR}$)

$\searrow$ $\qquad\qquad$ $\swarrow$

Weak König's Lemma (WKL_T)

$$\forall n^{\mathbb{N}} \exists u^{\mathbb{B}ool^*} \, (|u| = n \wedge u \in T) \rightarrow \exists \alpha^{\mathbb{N} \rightarrow \mathbb{B}ool} \, \forall n^{\mathbb{N}} \, T(\alpha_{|n})$$

A revisiting of the standard results

Generalised Axiom of Choice (GAC_{ABT})
(some extensionality theorem wrt ill-foundedness)

$\swarrow$ $\searrow$

Ultrafilter Theorem
($UF_{AT} \approx GAC_{A\mathbb{B}oolT}$)

Axiom of Dependent Choices
($DC_{BR} \approx GAC_{\mathbb{N}BR^+}$)

$\searrow$ $\swarrow$

Weak König's Lemma
($WKL_T =_{LEM} GAC_{\mathbb{N}\mathbb{B}oolT}$)

together with a dual picture with a Generalised Bar Induction

Towards the Ultrafilter theorem: Relaxing the sequentiality

Let A and B be domains. Let u now range over the set $(A \times B)^*$ of finite sequences of pairs of elements in A and B.

We write $dom(u) \in A^*$ for the sequence made of the left components of the pair u.

We say $(a, b) \in u$ if (a, b) is one of the components of u.

We write $u \leq v$ is u is included in v when seen as sets.

If $\alpha : A \to B$, we write $u \prec \alpha$ and say that u is a finite approximation of α if $\alpha(a) = b$ for all $(a, b) \in u$.

Let T be a predicate on $(A \times B)^*$. We write $\downarrow T$ and $\uparrow T$ to mean the following inner and outer closures with respect to $\leq$:

$$\begin{array}{l} u \in \downarrow T \triangleq \forall v \leq u \; v \in T \\ u \in \uparrow T \triangleq \exists v \leq u \; v \in T \end{array}$$

Relaxing the sequentiality (well-founded case)

Intensional view

We say that T is inductively A-B-barred if it is inductively A-B-barred from $\langle\rangle$ where the latter is defined by:

$$\frac{u \in \uparrow T}{T \text{ inductively } A\text{-}B\text{-barred from } u}$$

$$\frac{\exists a \in A\,(a \notin dom(u) \wedge \forall b \in B\ (T \text{ inductively } A\text{-}B\text{-barred from } (u \star (a,b))))}{T \text{ inductively } A\text{-}B\text{-barred from } u}$$

Extensional view

$$T \text{ is } A\text{-}B\text{-barred if } \forall \alpha : A \to B\ \exists u \prec \alpha\ u \in T$$

Relaxing the sequentiality (ill-founded case)

Intensional view

Dually, we say that T is coinductively A-B-choosable if it is coinductively A-B-choosable from $\langle\rangle$ where the latter is coinductively defined by:

$$\frac{u \in \downarrow T \qquad \forall a \in A\, (a \notin dom(u) \to \exists b \in B\ (T \text{ coind. } A\text{-}B\text{-choosable from } (u \star (a,b))))}{T \text{ coind. } A\text{-}B\text{-choosable from } u}$$

Extensional view

$$T \text{ is } A\text{-}B\text{-choosable if } \exists \alpha : A \to B\ \forall u \prec \alpha\ u \in T$$

This leads to the following generalisation

Generalised bar induction (GBI_{ABT})

T A-B-barred $\to$ T A-B-inductively barred

Generalised Tree-Based Dependent Choice ($GTDC_{ABT}$)

T coinductively A-B-choosable $\to$ T A-B-choosable

Justification of the generalisation

The generalisation is justified by the following results:

$$GBI_{\mathbb{N}BT} \iff BI_{BT}$$

$$GTDC_{\mathbb{N}BT} \iff TDC_{BT}$$

Ultrafilter Theorem

The specialisation to $\mathbb{B}\mathsf{ool}$ of the generalisation also captures the Ultrafilter Theorem.

Let $(\mathcal{B}, \vee, \wedge, \bot, \top, \neg, \vdash)$ be a Boolean algebra and F a filter on $\mathcal{B}$. We extend F on $(\mathcal{B} \times \mathbb{B}\mathsf{ool})^*$ by setting $u \in F^+$ if $(\bigwedge_{(b,0)\in u} \neg b) \vee (\bigvee_{(b,1)\in u} b) \in F$. We have:

$$GTDC_{\mathcal{B}\mathbb{B}\mathsf{ool}\neg(F^+)} \iff UF_{\mathcal{B},F}$$

Bad news suspected

Conjecture: $GTDC_{(\mathbb{B}\mathsf{ool}^{\mathbb{N}})\mathbb{N}T}$ is inconsistent!

Rationale: Trying to prove its equivalence with AC requires an A-intersection of non-empty subsets of $\mathbb{B}ool$ to be non-empty, which has no reason to hold in general. Moreover, standard algebraic statements equivalent to the axiom of choice such as Zorn's Lemma only asserts the existence of maximal (not necessarily "complete") ideals.

A suspected consistent generalisation

We define:

- $B_\perp \triangleq B + \{\perp\}$
- α *not T-extensible* at $a \triangleq \forall b \in B\, \exists u \prec \alpha\, (u \in T \wedge \neg(u \star (a, b) \in T))$.
- T maximally A-B-choosable $\triangleq$
$$\exists \alpha \in A \to B_\perp \begin{pmatrix} \forall a \in A\, (\alpha(a) = \perp \to \alpha \text{ not } T\text{-extensible at } a) \\ \wedge \forall u\, (u \prec \alpha \to u \in T) \end{pmatrix}$$

Generalised Axiom of Choice (GAC_{ABT})

T coinductively A-B-choosable $\to$ T maximally A-B-choosable

Computational contents

We know how to compute intuitionistically with DC using strong sums.

We know how to compute with BI using bar recursion.

We know how to compute with DC in the presence of classical reasoning using bar recursion and control operators.

We know how to compute with UF using ideas from the computational contents of completeness theorem.

Can bar recursion be extended to the generalised form? Presumably, the critical component is to deal with the non-decidability of equality on A.

Perspectives

Other variants of choice can probably be added to the picture.

Studying the principles together with their principle allow to see where non-linear reasoning is used. For instance, that WKL is essentially classical means that WFT is essentially intuitionistic in its formulation (on the contrary of BI or DC).

Understand to which versions of choice Diaconescu applies.

Explore other forms of choice and bar induction:

$\hookrightarrow$ hybrid forms such as J. Berger's C_{Fan}, i.e. to

$$\begin{array}{l} T \text{ coinductively choosable} \wedge U \text{ barred} \rightarrow \exists u\,(u \in T \wedge u \in U) \\ T \text{ choosable} \wedge U \text{ inductively barred} \rightarrow \exists u\,(u \in T \wedge u \in U) \end{array}$$

$\hookrightarrow$ well-ordering forms such as Zermelo's well-ordering theorem, open induction, U. Berger's update induction