Reverse mathematics of Gödel's completeness theorem

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Introduction

- Context: computing with proofs, even beyond intuitionistic logic, even possibly with side-effects (starting with classical logic)

- Completeness theorems: fundamental theorems connecting syntax and "semantics" (i.e. syntax from the meta-language)

  For instance, in the case of informative-enough models (Kripke/Beth models, phase-semantics/point-free-topology, Heyting/Boolean algebras, ...), completeness theorems replicate proofs of validity into proofs of derivability (cf e.g. Normalization-by-Evaluation)

- Gödel's completeness theorem: rich in its connection with standard axioms (Weak König's Lemma, Weak Fan Theorem, Ultrafilter Theorem, Markov's principle, ...)

- A large corpus of (often disconnected) results in the relative logical strength of axioms/theorems (so-called reverse mathematics): how to unify them?

- Knowing how to compute with Gödel's completeness, shall we be able to provide alternative ways to compute with the Weak Fan Theorem, Weak König's Lemma, Prime Ideal Theorem?

Outline

- **Reverse mathematics of Gödel's completeness theorem, in** PA_2, ZF, HA^2, HA_2, IZF, **...**

- Computing with Henkin's proof

- Tarski semantics as "direct-style" for Kripke semantics: towards a computation with side effects of Gödel's completeness

Classical reverse mathematics of the subsystems of second-order arithmetic

(the big five - Simpson 1999)

acronym	full name	canonical charact.	ordinal	f.o. fragment
RCA_0	Recursive Comprehension Axiom	Π^0_1-Π^0_1-Separation	ω^ω	PRA/IΣ_1
WKL_0	Weak König's Lemma	Σ^0_1-Σ^0_1-Separation	ω^ω	PRA/IΣ_1
ACA_0	Arithmetical Comprehension Axiom	Σ^0_1-Π^0_1-Separation	ϵ_0	PA
ATR_0	Arithmetical Transfinite Recursion	Σ^1_1-Σ^1_1-Separation	Γ_0	
$\Pi^1_1-\mathsf{CA}_0$	Π^1_1 Comprehension Axiom	Σ^1_1-Π^1_1-Separation		

A typical result in this context (Simpson):

$\mathsf{RCA}_0 \vdash$ *Gödel's completeness theorem = Weak König's Lemma*

Moreover:

$\mathsf{RCA}_0 \vdash$ *full König's Lemma* $= \mathsf{ACA}_0$

Classical reverse mathematics in set theory

Typically about the axiom of choice (e.g. Jech 1973, Howard-Rubin 1998)

$$\begin{array}{ccccc}
\left\{\begin{array}{l}\text{Axiom of choice}\\ \text{Zorn's lemma}\\ \text{Well-ordering theorem}\end{array}\right\} & & \Rightarrow & & \left\{\begin{array}{l}\text{Prime Ideal Theorem}\\ \text{Ultrafilter Theorem}\end{array}\right\} \\
\Downarrow & & & & \Downarrow \\
\left\{\begin{array}{l}\text{Dependent choices}\\ \text{Bar induction}\end{array}\right\} & \Rightarrow \text{Countable choice} & \Rightarrow & & \text{Countable choice over a two-element set} \\
 & & & & \Downarrow \\
 & & & & \left\{\begin{array}{l}\text{Countable choice over Booleans}\\ \text{Weak König's lemma}\\ \text{Weak Fan theorem}\end{array}\right\}
\end{array}$$

Typical results in this context:

Henkin (1954): **ZF** $\vdash$ *Gödel's completeness theorem = Prime Ideal Theorem*

Espindola (2016): **ZF** $\vdash$ *completeness wrt Kripke models = Prime Ideal Theorem*

McCarty (2004): **IZF** $\vdash$ *Gödel's completeness* $\Rightarrow$ *EM*

Constructive reverse mathematics

Typically done within HA^2 with weak choice principles (Veldman's **BIM**, Kleene-Vesley's **WKV**, Kreisel-Troelstra's **EL**, ...)

Many various results, sometimes looking contradictory:

Gödel (1957), Kreisel (1962): $\mathsf{HA}_2 \vdash$ completeness wrt Tarski semantics $\Rightarrow$ Markov's principle

Friedman (1975): $\mathsf{HA}^2 \vdash$ completeness wrt Beth models (but... fallible models)

Veldman (1976): $\mathsf{HA}^2 + \mathsf{WFT} \vdash$ completeness wrt Kripke models (but... with exploding nodes)

Krivine (1996): $\mathsf{HA}_2 \vdash$ Gödel's completeness (but... finite theory and only $\Rightarrow$, $\forall$)

Berardi (1999): $\mathsf{HA}_2 \nvdash$ Gödel's completeness if $\vee$ or $\perp$ have their Tarskian semantics

Berardi-Valentini (2004): $\mathsf{HA}_2 \vdash$ Prime Ideal Theorem over a countable Boolean algebra (but with a definition of prime ideal avoiding $\vee$)

Loeb (2005): **WKV** $\vdash$ Weak Fan Theorem $=$ Fan Theorem (!!)
$=$ Gödel's completeness (but...)

Espindola (2016): **IZF** $\vdash$ Gödel's completeness $=$ EM $+$ Prime Ideal Theorem (but...)

Different formulation of Gödel's completeness

- For arbitrary theories, *valid* implies *provable*

$$\forall \mathcal{T}\, [\forall \mathcal{M} (\, \vDash_{\mathcal{M}} \mathcal{T} \Rightarrow \vDash_{\mathcal{M}} A) \Rightarrow \mathcal{T} \vdash A]$$

$\hookrightarrow$ version considered by Espindola (2016)

- For recursively enumerable theories, *valid* implies *provable*

$$\forall \mathcal{T}\, [\forall \mathcal{M} (\, \vDash_{\mathcal{M}} \mathcal{T} \Rightarrow \vDash_{\mathcal{M}} A) \Rightarrow \mathcal{T} \vdash A]$$

$\hookrightarrow$ equivalent to (a weak form of) Weak Fan Theorem in the presence of $\Rightarrow$, $\wedge$, $\forall$

$\hookrightarrow$ equivalent to (the usual - strong - form of) Weak Fan Theorem in the presence of $\vee$

$\hookrightarrow$ additionally requires Markov's principe in the presence of $\bot$

- For (recursively enumerable) theories, *consistent* implies *has a model*

$$\forall \mathcal{T}\, [\mathcal{T} \nvdash \bot \Rightarrow \exists \mathcal{M}\ \vDash_{\mathcal{M}} \mathcal{T}]$$

Markov's Principle no longer needed for the case of $\bot$

Different formulation of Gödel's completeness (continued)

- Weak form of *valid* implies *provable*

$$\forall \Gamma\, [\forall \mathcal{M} (\models_{\mathcal{M}} \Gamma \Rightarrow \models_{\mathcal{M}} A) \Rightarrow \Gamma \vdash A]$$

- *Provable* or *has a model*

$$\forall \mathcal{T}\, [\mathcal{T} \vdash A \ \vee \ \exists \mathcal{M} \models_{\mathcal{M}} \mathcal{T} \wedge \neg A]$$

$\hookrightarrow$ strongly classical

Note: formal proofs of the above statement not all yet written.

Tarski semantics vs 2-valued semantics

From an intuitionistic reverse math. point of view, it matters how a model is defined:

- a set of propositions?

  i.e. $\mathcal{M} : \mathcal{F}orm \rightarrow Prop$

- a functional relation mapping propositions to Booleans?

  i.e. $\mathcal{M} : \Sigma R : \mathcal{F}orm \times \mathbb{B} \rightarrow Prop. \forall A \, \exists! b \, R\,(A, b)$

- a function mapping propositions to Booleans?

  i.e. $\mathcal{M} : \mathcal{F}orm \rightarrow \mathbb{B}$

Tarski semantics vs 2-valued semantics

Obviously:

$$\mathcal{F}orm \to \mathbb{B}$$

$$\Downarrow$$

$$\Sigma R : \mathcal{F}orm \times \mathbb{B} \to Prop.\, \forall A\, \exists! b\, R\,(A, b)$$

$$\Downarrow$$

$$\mathcal{F}orm \to Prop$$

Map $f : \mathcal{F}orm \to \mathbb{B}$ to $R(A, b) \triangleq (f(A) = b)$ which is trivially functional

Map $R : \mathcal{F}orm \times \mathbb{B} \to Prop$ to $X(A) \triangleq R(A, \textsf{true})$

Tarski semantics vs 2-valued semantics

And also:

$$\mathcal{F}orm \to \mathbb{B}$$

$$\mathsf{AC!}_{\mathbb{N},\mathbb{B}} \ \Uparrow$$

$$\Sigma R : \mathcal{F}orm \times \mathbb{B} \to Prop . \forall A \, \exists! b \, R\,(A, b)$$

$$\mathsf{EM} \Uparrow$$

$$\mathcal{F}orm \to Prop$$

Map $X : \mathcal{F}orm \to Prop$ to $R(A, b) \triangleq (b = \mathsf{true} \Leftrightarrow X(A))$, this is functional by EM

Map $\forall A \, \exists! b \, R\,(A, b)$ to a function by unique choice.

On the three ways to formalize subsets

The three different styles applies also to state the Weak König's Lemma, Weak Fan Theorem, Boolean Prime Ideal, ...

Intuitionistic reverse mathematics favor the *functional* form (e.g. Veldman)

Classical reverse mathematics favor the *functional relation* form (e.g. Simpson)

The *predicate* form is the easiest to compute with in the case of the above axioms/theorems

Three corresponding forms of Weak Fan Theorem

(contraposition of Weak König's Lemma / bar induction on binary trees)

Let T be an arbitrary predicate on $\mathbb{B}^*$ (finite sequences of Booleans)

$$\begin{array}{ll}
\mathsf{WFT}_{fun} & \triangleq \forall f\, \exists n\, T(f_{|n}) \;\Rightarrow\; \exists N\, \forall l\, (|l| = N \Rightarrow \exists l' \subset l\, T(l')) \\[1em]
\mathsf{WFT}_{fun-rel} & \triangleq \forall R\, \exists n\, \exists l\, \approx_n R \wedge T(l) \;\Rightarrow\; \exists N\, \forall l\, (|l| = N \Rightarrow \exists l' \subset l\, T(l')) \\[1em]
\mathsf{WFT}_{pred} & \triangleq \forall X\, \exists n\, \exists l\, \approx_n X \wedge T(l) \;\Rightarrow\; \exists N\, \forall l\, (|l| = N \Rightarrow \exists l' \subset l\, T(l'))
\end{array}$$

where:

$$\frac{}{\epsilon \approx_0 X} \qquad \frac{l \approx_n X \quad X(n)}{l \cdot \mathsf{true} \approx_{n+1} X} \qquad \frac{l \approx_n X \quad \neg X(n)}{l \cdot \mathsf{false} \approx_{n+1} X}$$

$$\frac{}{\epsilon \approx_0 R} \qquad \frac{l \approx_n R \quad R(n,b)}{l \cdot b \approx_{n+1} R} \qquad\qquad \begin{array}{ll} f_{|0} & \triangleq \epsilon \\ f_{|n+1} & \triangleq f_{|n} \cdot f(n) \end{array}$$

Note: We do not care here about the logical complexity of T

Three forms of Weak Fan Theorem

Thus we have:

$$\mathsf{WFT}_{pred} \overset{\mathsf{EM}}{\Rightarrow} \mathsf{WFT}_{fun-rel} \overset{\mathsf{AC!}_{\mathbb{N},\mathbb{B}}}{\Rightarrow} \mathsf{WFT}_{fun}$$

$$\mathsf{WFT}_{fun} \Rightarrow \mathsf{WFT}_{fun-rel} \Rightarrow \mathsf{WFT}_{pred}$$

WFT_{fun}, considered in intuitionistic reverse mathematics, is equivalent to the full Fan Theorem on finite (non-necessarily binary) "trees"

$\mathsf{WFT}_{fun-rel}$ and WFT_{pred}, both equivalent in classical reverse mathematics, are not equivalent to the corresponding formulation of the full Fan Theorem

Intuitionistically, WFT_{pred} is enough to prove completeness in the presence of $\Rightarrow$, $\wedge$, $\forall$ (over recursively enumerable theories)

$\mathsf{WFT}_{fun-rel}$ is needed for $\vee$

On the respective force of the different formulations of Weak Fan Theorem

After Berger (2009) who isolated the classical part L_{fan} ("in a binary tree with at most one infinite branch, we can decide whether it is the left or right subtree which is infinite) and choice part C_{fan} of WFT:

Conjecture, for S a class of formula:

$$\begin{array}{ccc} C_{fan-pred}(S) = \mathsf{WFT}_{pred}(S) & \stackrel{L_{fan}(S)}{\Rightarrow} & \mathsf{WFT}_{fun-rel}(S) \\ \Downarrow \mathsf{AC!} & & \Downarrow \mathsf{AC!} \\ C_{fan-fun}(S) & \stackrel{L_{fan}(S)}{\Rightarrow} & \mathsf{WFT}_{fun}(S) \end{array}$$

Similarly, following Ishihara (2005), conjecture:

$$\begin{array}{ccc} C_{\mathsf{WKL}-pred}(S) = \mathsf{WKL}_{pred}(S) & \stackrel{\mathsf{WEM}(\Sigma S)}{\Rightarrow} & \mathsf{WKL}_{fun-rel}(S) \\ \Downarrow \mathsf{AC!} & & \Downarrow \mathsf{AC!} \\ C_{\mathsf{WKL}-fun}(S) & \stackrel{\mathsf{WEM}(\Sigma S)}{\Rightarrow} & \mathsf{WKL}_{fun}(S) \end{array}$$

with $\mathsf{WEM}(S) \triangleq \neg(A \wedge B) \Rightarrow \neg A \vee \neg B$ for $A, B \in S$

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The statement of completeness
(weak form, restricted to the negative fragment)

$$
\begin{array}{l}
t \in \mathcal{T} ::= x \mid f t_1 \ldots t_{a_f} \\
A, B \in \mathcal{F} ::= P t_1 \ldots t_{a_P} \mid \dot{\bot} \mid A \dot{\rightarrow} B \mid \dot{\forall} x\, A
\end{array}
$$

A model is a triple $(\mathcal{M}_D, \mathcal{M}(f) \in \mathcal{M}_D^{a_f} \Rightarrow \mathcal{M}_D, \mathcal{M}(P) \in \mathcal{P}(\mathcal{M}_D^{a_P}))$. Truth in $\mathcal{M}$ is defined recursively:

$$
\begin{array}{lcl}
[\![x]\!]^\sigma_{\mathcal{M}} & \triangleq & \sigma(x) \\
[\![f t_1 \ldots t_{a_f}]\!]^\sigma_{\mathcal{M}} & \triangleq & \mathcal{M}(f)[\![t_1]\!]^\sigma_{\mathcal{M}} \ldots [\![t_{a_f}]\!]^\sigma_{\mathcal{M}} \\
[\![P t_1 \ldots t_{a_f}]\!]^\sigma_{\mathcal{M}} & \triangleq & \mathcal{M}(P)[\![t_1]\!]^\sigma_{\mathcal{M}} \ldots [\![t_{a_P}]\!]^\sigma_{\mathcal{M}} \\
[\![\dot{\bot}]\!]^\sigma_{\mathcal{M}} & \triangleq & \bot \\
[\![A \dot{\rightarrow} B]\!]^\sigma_{\mathcal{M}} & \triangleq & [\![A]\!]^\sigma_{\mathcal{M}} \Rightarrow [\![B]\!]^\sigma_{\mathcal{M}} \\
[\![\forall x\, A]\!]^\sigma_{\mathcal{M}} & \triangleq & \forall t \in \mathcal{M}_D\, [\![A]\!]^{\sigma[x \leftarrow t]}_{\mathcal{M}}
\end{array}
$$

A model is classical, written $Class(\mathcal{M})$ if for each A and σ, $[\![\dot{\neg}\dot{\neg} A]\!]^\sigma_{\mathcal{M}} \Rightarrow [\![A]\!]^\sigma_{\mathcal{M}}$.

The completeness statement : $\forall A\, (\forall \mathcal{M} \forall \sigma\, Class(\mathcal{M}) \Rightarrow [\![A]\!]^\sigma_{\mathcal{M}}) \Rightarrow\ [\vdash A]$

The proof (usual presentation)

To prove $\vdash A_0$, prove instead $\dot\neg A_0 \vdash \dot\bot$.

Reason by contradiction and assume $(\dot\neg A_0 \vdash \dot\bot) \Rightarrow \bot$, i.e. that the context $\Gamma_0 \triangleq \dot\neg A_0$ is consistent.

For an enumeration $A_1, A_3, A_5, \ldots$ of all non-universal formulas and an enumeration $\dot\forall x\, A_2, \dot\forall x\, A_4, \dot\forall x\, A_6, \ldots$ of all universal formulas, classically build

- $\Gamma_{2n+1} \triangleq \Gamma_{2n}$ if $\Gamma_{2n}, A_{2n+1} \vdash \dot\bot$
- $\Gamma_{2n+1} \triangleq \Gamma_{2n}, A_{2n+1}$ otherwise
- $\Gamma_{2n+2} \triangleq \Gamma_{2n+1}, (A_{2n+2}[x_n/x] \dot\to \dot\forall x\, A_{2n+2})$ if $\Gamma_n, \dot\forall x\, A_{2n+2} \vdash \dot\bot$
- $\Gamma_{2n+2} \triangleq \Gamma_{2n+1}, (A_{2n+2}[x_n/x] \dot\to \dot\forall x\, A_{2n+2}), \dot\forall x\, A_{2n+2}$ otherwise

where the formulas $A_{2n+2}[x_n/x] \dot\to \dot\forall x\, A_{2n+2}$, for x_n taken fresh in Γ_{2n+1} are called Henkin axioms.

This construction propagates consistency from Γ_0 to Γ_n.

The proof (usual presentation), continued

Build the infinite theory $\mathcal{T} \triangleq \cup_n \Gamma_n$.

Under the initial assumption that $\vdash A_0$ is contradictory, one gets that $\mathcal{T}$ is consistent.

Define a syntactic model $\mathcal{M}_0$ by $\mathcal{D} \triangleq \mathcal{T}$, $\mathcal{M}(f)(t_1, \ldots, t_{a_f}) \triangleq f(t_1, \ldots, t_{a_f})$ and $\mathcal{M}(P)(t_1, \ldots, t_{a_P}) \triangleq P(t_1, \ldots, t_{a_P}) \in \mathcal{T}$.

One can prove by induction on A that $[\![A]\!]_{\mathcal{M}_0}$ iff $A \in \mathcal{T}$.

The model is complete in the sense that either $A \in \mathcal{T}$ or $\dot{\neg} A \in \mathcal{T}$, and hence satisfy $Class(\mathcal{M}_0)$.

By validity of A_0, get $[\![A_0]\!]_{\mathcal{M}_0}$, hence $A_0 \in \mathcal{T}$ hence $\mathcal{T} \vdash \dot{\bot}$, a contradiction.

The proof (turned positively)

Let $\lceil A \rceil$ and ϕ form a Gödel's numbering of formulas such that $\lceil \phi(n) \rceil = n$. Let x_n be a variable fresh in $\phi(0), \ldots, \phi(n)$. Henkin axioms at step n are defined by taking Θ_0 to be empty and Θ_{n+1} to be Θ_n unless $\phi(n) = \dot{\forall} x\, A$ in which case it is $A[x_n/x] \Rightarrow \forall x\, A, \Theta_n$. Let A_0 be the formula we expect a proof of.

Let F_n (virtually) denotes the countermodel built at step n. We define $A \in F_\omega$ to mean $\exists n \exists \Gamma \subset F_n\, [\Theta_n, \Gamma \vdash A]$ ("A gets provable at some step of the construction of a context equiconsistent to $\dot{\neg} A_0$") where $\Gamma \subset F_n$ is formally defined inductively:

$$\frac{}{\dot{\neg} A_0 \subset F_0}\ I_0 \qquad\qquad \frac{\Gamma \subset F_n}{\Gamma \subset F_{n+1}}\ I_S$$

$$\frac{\Gamma \subset F_n \qquad \forall \Gamma' \subset F_n\ \ [\Theta_n, \Gamma', \{A\}_n \vdash \dot{\bot}] \Rightarrow \bot}{\Gamma, \{A\}_n \subset F_{n+1}}\ I_n$$

where $\{A\}_n$ is $A[x_n/x]$ if $\phi(n) = \dot{\forall} x\, A$ and $\phi(n)$ otherwise.

The (syntactic) model $\mathcal{M}_0$ is defined by $\mathcal{D} \triangleq \mathcal{T}$, $\mathcal{M}(f)(t_1, \ldots, t_{a_f}) \triangleq f(t_1, \ldots, t_{a_f})$ and $\mathcal{M}(P)(t_1, \ldots, t_{a_P}) \triangleq P(t_1, \ldots, t_{a_P}) \in F_\omega$.

The object language

We assume given a (non-minimal) set of appropriate object language constructions:

$\dot{\mathtt{ax}}_i : [\Gamma, A, \Gamma' \vdash A]$ (for Γ' of length i)

$\dot{\mathtt{ax}}'_i : [\Gamma, A, \Gamma' \vdash A]$ (for Γ of length i)

$\dot{\mathtt{dn}} : [\Gamma \vdash \dot{\neg}\dot{\neg} A] \longrightarrow [\Gamma \vdash A]$

$\dot{\mathtt{abs}} : [\Gamma, A \vdash B] \longrightarrow [\Gamma \vdash A \dot{\to} B]$

$\dot{\mathtt{app}}^{\dot{\Rightarrow}} : [\Gamma \vdash A \dot{\to} B] \longrightarrow [\Gamma' \vdash A] \longrightarrow [\Gamma \cup \Gamma' \vdash B]$

$\dot{\mathtt{drinker}}_n : [A[x_n/x] \dot{\to} \dot{\forall} x\, A, \Gamma \vdash \dot{\bot}] \longrightarrow [\Gamma \vdash \dot{\bot}]$ if $\phi(n+1) = \dot{\forall} x A$ and x_n not in $\dot{\forall} x\, A, \Gamma$

$\dot{\mathtt{drinker}}_n : [\Gamma \vdash \bot] \longrightarrow [\Gamma \vdash \dot{\bot}]$ otherwise

$\dot{\mathtt{app}}^{\dot{\forall}} : [\Gamma \vdash \dot{\forall} x\, A(x)] \longrightarrow \forall t \in \mathcal{T}\, [\Gamma \vdash A(t)]$

$\pi_1^{\dot{\to}} : [\Gamma, A \dot{\to} B \vdash \dot{\bot}] \longrightarrow [\Gamma \vdash A]$

$\pi_2^{\dot{\to}} : [\Gamma, A \dot{\to} B \vdash \dot{\bot}] \longrightarrow [\Gamma \vdash \dot{\neg} B]$

$\dot{\mathtt{efq}} : [\Gamma \vdash \dot{\bot}] \longrightarrow [\Gamma \vdash A]$

The core of the proof

$$
\begin{array}{llll}
\downarrow_A & : A \in \mathcal{M} & \to & A \in F_\omega \\
\downarrow_{P(\vec{t})} & m & \triangleq & m \\
\downarrow_{\dot\bot} & m & \triangleq & \mathtt{e\dot{f}q}\, m \\
\downarrow_{A \dot\to B} & m & \triangleq & (n, (\dot\neg A_0, A \dot\to B), \\
 & & & \quad I_n(\mathtt{inj}_n, (\Gamma, f, p) \mapsto \begin{array}{l} \mathtt{dest}\ \downarrow_B (m(\uparrow_A (n, \Gamma, f, \pi_1^{\dot\to}\, p)))\ \mathsf{as}\ (n', \Gamma', f', p') \\ \mathtt{in\ flush}^{\Gamma\cup\Gamma'}_{max(n,n')}(\mathtt{join}^{\Gamma\Gamma'}_{nn'}(f, f'), \mathsf{a\dot{p}p}^{\Rightarrow}(\pi_2^{\dot\to}\, p, p')) \end{array} \\
 & & & \quad \mathtt{a\dot{x}}_1) \hfill \textit{where } n = \lceil A \dot\to B \rceil \\
\downarrow_{\dot\forall x\, A} & m & \triangleq & \mathtt{dest}\ \downarrow_{A[x_n/x]} (m\, x_n)\ \mathsf{as}\ (n', \Gamma', f', p') \\
 & & & \mathtt{in}\ (max(n, n'), \Gamma', \mathtt{join}^{(\dot\neg A_0)\Gamma'}_{nn'}(\mathtt{inj}_n, f'), \mathsf{a\dot{p}p}^{\Rightarrow}(\mathtt{a\dot{x}}'_0, p')) \\
 & & & \hfill \textit{where } n = \lceil \dot\forall x\, A \rceil
\end{array}
$$

$$
\begin{array}{llll}
\uparrow_A & : A \in F_\omega & \to & A \in \mathcal{M} \\
\uparrow_{P(\vec{t})} & (n, \Gamma, f, p) & \triangleq & (n, \Gamma, f, p) \\
\uparrow_{\dot\bot} & (n, \Gamma, f, p) & \triangleq & \mathtt{flush}^{\Gamma}_n(f, p) \\
\uparrow_{A \dot\to B} & (n, \Gamma, f, p) & \triangleq & m \mapsto \begin{array}{l} \mathtt{dest}\ \downarrow_A m\ \mathsf{as}\ (n', \Gamma', f', p') \\ \mathtt{in}\ \uparrow_B (max(n, n'), \Gamma \cup \Gamma', \mathtt{join}^{\Gamma\Gamma'}_{nn'}(f, f'), \mathsf{a\dot{p}p}^{\Rightarrow}(p, p')) \end{array} \\
\uparrow_{\dot\forall x\, A} & (n, \Gamma, f, p) & \triangleq & t \mapsto \uparrow_{A[t/x]} (n, \Gamma, f, \mathsf{a\dot{p}p}^{\forall}(p, t))
\end{array}
$$

Auxiliary lemmas

$$\begin{array}{llll}
\mathtt{flush}^{\Gamma}_{n} & : \Gamma \subset F_n \wedge [\Theta_n, \Gamma \vdash \dot{\bot}] & \longrightarrow & \bot \\
\mathtt{flush}^{\Gamma}_{0} & (\mathtt{I}_0, p) & \triangleq & \mathtt{throw}_{\alpha_0}\, p \\
\mathtt{flush}^{\Gamma}_{n+1} & (\mathtt{I}_\mathtt{S} f, p) & \triangleq & \mathtt{flush}^{\Gamma}_{n}(f, \mathtt{dr\dot{i}nker}_n\, p) \\
\mathtt{flush}^{\Gamma A}_{n+1} & (\mathtt{I}_\mathtt{n}(f, k), p) & \triangleq & k\ \Gamma\ f\ p
\end{array}$$

$$\begin{array}{llllll}
\mathtt{join}^{\Gamma_1\Gamma_2}_{n_1 n_2} & : \Gamma_1 \subset F_{n_1} & \wedge\ \Gamma_2 \subset F_{n_2} & \longrightarrow & \Gamma_1 \cup \Gamma_2 \subset F_{max(n_1,n_2)} & \\
\mathtt{join}^{\dot{\neg} A_0 \dot{\neg} A_0}_{00} & \mathtt{I}_0 & \mathtt{I}_0 & \triangleq & \mathtt{I}_0 & \\
\mathtt{join}^{(\Gamma_1 A)(\Gamma_2 A)}_{(n+1)(n+1)} & \mathtt{I}_\mathtt{n}(f_1, k_1) & \mathtt{I}_\mathtt{n}(f_2, k_2) & \triangleq & \mathtt{I}_\mathtt{n}(\mathtt{join}^{\Gamma_1\Gamma_2}_{nn} f_1 f_2, k_1) & \\
\mathtt{join}^{(\Gamma_1 A)\Gamma_2}_{(n+1)(n+1)} & \mathtt{I}_\mathtt{n}(f_1, k_1) & \mathtt{I}_\mathtt{S} f_2 & \triangleq & \mathtt{I}_\mathtt{n}(\mathtt{join}^{\Gamma_1\Gamma_2}_{nn} f_1 f_2, k_1) & \\
\mathtt{join}^{\Gamma_1(\Gamma_2 A)}_{(n+1)(n+1)} & \mathtt{I}_\mathtt{S} f_1 & \mathtt{I}_\mathtt{n}(f_2, k_2) & \triangleq & \mathtt{I}_\mathtt{n}(\mathtt{join}^{\Gamma_1\Gamma_2}_{nn} f_1 f_2, k_2) & \\
\mathtt{join}^{\Gamma_1\Gamma_2}_{(n+1)(n+1)} & \mathtt{I}_\mathtt{S} f_1 & \mathtt{I}_\mathtt{S} f_2 & \triangleq & \mathtt{I}_\mathtt{S}(\mathtt{join}^{\Gamma_1\Gamma_2}_{nn} f_1 f_2) & \\
\mathtt{join}^{\Gamma_1\Gamma_2}_{n_1 n_2} & \mathtt{I}_\mathtt{S} f_1 & f_2 & \triangleq & \mathtt{I}_\mathtt{S}(\mathtt{join}^{\Gamma_1\Gamma_2}_{n_1' n_2} f_1 f_2) & \text{if } n_1 = n_1' + 1 > n_2 \\
\mathtt{join}^{(\Gamma_1 A_1)\Gamma_2}_{n_1 n_2} & \mathtt{I}_{\mathtt{n}_1'}(f_1, k_1) & f_2 & \triangleq & \mathtt{I}_{\mathtt{n}_1'}(\mathtt{join}^{\Gamma_1\Gamma_2}_{n_1' n_2} f_1 f_2, k_1) & \text{if } n_1 = n_1' + 1 > n_2 \\
\mathtt{join}^{\Gamma_1\Gamma_2}_{n_1 n_2} & f_1 & \mathtt{I}_\mathtt{S} f_2 & \triangleq & \mathtt{I}_\mathtt{S}(\mathtt{join}^{\Gamma_1\Gamma_2}_{n_1 n_2'} f_1 f_2) & \text{if } n_1 < n_2' + 1 = n_2 \\
\mathtt{join}^{\Gamma_1(\Gamma_2 A_2)}_{n_1 n_2} & f_1 & \mathtt{I}_{\mathtt{n}_2'}(f_2, k_2) & \triangleq & \mathtt{I}_{\mathtt{n}_2'}(\mathtt{join}^{\Gamma_1\Gamma_2}_{n_1 n_2'} f_1 f_2, k_2) & \text{if } n_1 < n_2' + 1 = n_2
\end{array}$$

$$\begin{array}{lll}
\mathtt{inj}_n & : & (\dot{\neg} A_0) \subset F_n \\
\mathtt{inj}_0 & \triangleq & \mathtt{I}_0 \\
\mathtt{inj}_{n+1} & \triangleq & \mathtt{I}_\mathtt{S}(\mathtt{inj}_n)
\end{array}$$

Final weak completeness result

$$\begin{array}{llll} \mathtt{class}_0 & : (\dot{\neg}\dot{\neg}A) \in \mathcal{M}_0 & \longrightarrow & A \in \mathcal{M}_0 \\ \mathtt{class}_0 & m & \triangleq & \uparrow_A (\mathtt{dest} \downarrow_{\dot{\neg}\dot{\neg}A} m \,\mathtt{as}\, (n, \Gamma, f, p) \,\mathtt{in}\, (n, \Gamma, f, \dot{\mathtt{dn}} p)) \end{array}$$

$$\begin{array}{llll} \mathtt{compl}_{\mathtt{A}_0} & : (\forall \mathcal{M} \forall \sigma\, Class(\mathcal{M}) \Rightarrow [\![A_0]\!]^{\sigma}_{\mathcal{M}}) & \longrightarrow & \vdash A_0 \\ \mathtt{compl}_{\mathtt{A}_0} & \psi & \triangleq & \dot{\mathtt{dn}}(\dot{\mathtt{abs}}(\mathtt{catch}_{\alpha_0} \begin{array}{l} \mathtt{dest} \downarrow_{A_0} (\psi\, \mathcal{M}_0\, \mathtt{id}\, \mathtt{class}_0) \,\mathtt{as}\, (n, \Gamma, f, p) \\ \mathtt{in}\, \dot{\mathtt{efq}}\, \mathtt{flush}^{\Gamma}_{n}(f, \dot{\mathtt{app}}^{\Rightarrow}(\dot{\mathtt{ax}}_{|\Gamma|-1}, p)) \end{array})) \end{array}$$

Outline

- Reverse mathematics of Gödel's completeness theorem, in PA_2, ZF, HA^2, HA_2, IZF, ...
- Computing with Henkin's proof
- **Tarski semantics as "direct-style" for Kripke semantics: towards a computation with side effects of Gödel's completeness**

Preliminary I: Soundness, completeness and semantic normalisation

- (strong completeness $\circ$ soundness) gives cut-elimination
- For "rich-enough" semantics (Kripke, Beth, point-free topology, phase semantics, ...) can be turned into semantic normalisation (Berger-Schwichtenberg 1991, C. Coquand 2002, ...), also related to type-directed partial evaluation (Danvy 1996, ...) following the same proof pattern as in reducibility proofs:
  - adequacy/soundness: $\mathcal{T} \vdash A$ implies ($[\![\mathcal{T}]\!]$ implies $[\![A]\!]$) (for some semantics)
    $\hookrightarrow$ proved by induction on proofs
  - escape lemma/completeness: mutually proving
    reflection ($\uparrow$) : $\mathcal{T} \vdash_{neutral} A$ implies $[\![A]\!]$
    reification ($\downarrow$) : $[\![A]\!]$ implies $\mathcal{T} \vdash_{nf} A$
    $\hookrightarrow$ by mutual induction on A

Can we do the same w.r.t. Tarskian semantics?

Preliminary II: Proving with side effects

- Classical logic seen as a side effect:
  - Direct style = a control operator (e.g. c*c* of type Peirce's law) [Griffin 90]
  - Indirect style = continuation-passing-style/double-negation translation within intuitionistic logic ($K(A) \triangleq \neg\neg A$ and $(A \Rightarrow B)^* \triangleq A^* \Rightarrow K(B^*)$, etc.)
- This part of the talk:
  - Interpreting Kripke forcing translation as indirect style for what is in direct style a monotonic memory update
  - Applying this to obtain a proof with side-effect of Gödel's completeness theorem as direct-style presentation of a proof of completeness w.r.t. Kripke semantics

Kripke forcing translation

Let $\geq$ be a partial order. A key clause of Kripke forcing is the interpretation of implication:

$$w \Vdash A \Rightarrow B \triangleq \forall w' \geq w \, [(w' \Vdash A) \Rightarrow (w' \Vdash B)]$$

The transformation

$$\Box_w A(w) \triangleq \forall w' \geq w \, A(w')$$

can be seen as a dependent environment-passing-style translation, i.e. as indirect style for a monotonic memory update effect.

Environment-passing-translation

$$
\begin{array}{lcl}
E(A) & \triangleq & W \Rightarrow A \\
\\
(A \Rightarrow B)^* & \triangleq & A^* \Rightarrow E(B^*) \\
X^* & \triangleq & X \\
\\
(\Gamma \vdash A)^* & \triangleq & \Gamma^* \vdash E(A^*) \\
\\
\eta & : & A \Rightarrow E(A) \\
\eta x & \triangleq & \lambda w.x \\
\texttt{>>=} & : & E(A) \Rightarrow (A \Rightarrow E(B)) \Rightarrow E(B) \\
u \ \texttt{>>=} \ t & \triangleq & \lambda w.t(uw)w \\
\\
x^* & \triangleq & \eta x \\
(\lambda x.t)^* & \triangleq & \eta \lambda x.t^* \\
(tu)^* & \triangleq & t^* \ \texttt{>>=} \ \lambda f.(u^* \ \texttt{>>=} \ f) \\
\\
\texttt{w}^* & \triangleq & \lambda w.w \\
(\texttt{update w} := t \ \texttt{in} \ u)^* & \triangleq & t^* \ \texttt{>>=} \ \lambda w.u^* w
\end{array}
$$

Direct-style for Kripke forcing

A rule for initialising the use of Kripke forcing:

$$\frac{\begin{array}{l}\Gamma, [b : x \geq t] \vdash q : T(x) \\ \Gamma \vdash r : \mathit{refl}\ \geq \\ \Gamma \vdash s : \mathit{trans}\ \geq \\ x \text{ fresh in } \Gamma \text{ and } T(t)\end{array}}{\Gamma \vdash \mathtt{set}\, x := t\, \mathtt{as}\, b/_{(r,s)}\, \mathtt{in}\, q : T(t)}\ \text{SETEFF}$$

A rule for updating:

$$\frac{\begin{array}{l}\Gamma, [b : x \geq t(x')] \vdash q : T(x) \\ \Gamma \vdash r : t(x') \geq x' \\ [x \geq u] \in \Gamma \text{ for some } u \\ x' \text{ fresh in } \Gamma\end{array}}{\Gamma \vdash \mathtt{update}\, x := t(x)\, \mathtt{of}\, x'\, \mathtt{as}\, b\, \mathtt{by}\, r\, \mathtt{in}\, q : T(t(x))}\ \text{UPDATE}$$

where we wrote T, U for $\dot{\rightarrow}$-$\dot{\forall}$-free formulas (= intuitively Σ^0_1-formulas = base types)

Gödel's completeness

Object language

We consider here the negative fragment of predicate logic as an object language (we consider $\dot{\bot}$ to be an arbitrary atom and abbreviate $\dot{\neg} A \triangleq A \dot{\to} \dot{\bot}$).

$$\begin{array}{lll} t & \triangleq & x \mid f(t_1, ..., t_n) \\ F, G & \triangleq & \dot{\bot} \mid \dot{P}(t_1, ..., t_n) \mid F \dot{\to} G \mid \dot{\forall} x\, F \\ \Gamma & \triangleq & \epsilon \mid \Gamma, F \end{array}$$

We take the following inference rules:

$$\begin{array}{ll} \dot{\mathtt{Ax}}^{\Gamma,F,\Gamma'} & : (\Gamma, F \subset \Gamma') \Rightarrow (\Gamma' \vdash F) \\ \dot{\mathtt{App}}_{\Rightarrow}^{\Gamma,F,G} & : (\Gamma \vdash F \dot{\to} G) \Rightarrow (\Gamma \vdash F) \Rightarrow (\Gamma \vdash G) \\ \dot{\mathtt{Abs}}_{\Rightarrow}^{\Gamma,F,G} & : (\Gamma, F \vdash G) \Rightarrow (\Gamma \vdash F \dot{\to} G) \\ \dot{\mathtt{Abs}}_{\forall}^{\Gamma,x,F} & : (\Gamma \vdash F) \Rightarrow (x \notin FV(\Gamma)) \Rightarrow (\Gamma \vdash \dot{\forall} x\, F) \\ \dot{\mathtt{App}}_{\forall}^{\Gamma,x,t,F} & : (\Gamma \vdash \dot{\forall} x\, F) \Rightarrow (\Gamma \vdash F[t/x]) \end{array}$$

Moreover, the following is admissible:

$$\dot{\mathtt{weak}}_{\Gamma,F}^{\Gamma'} \; : (\Gamma \subset \Gamma') \Rightarrow (\Gamma \vdash F) \Rightarrow (\Gamma' \vdash F)$$

We shall also write r_F^{Γ} for a proof of $\Gamma \subset (\Gamma, F)$.

Tarskian models

A Tarskian model $\mathcal{M}$ is made of a domain $\mathcal{D}_\mathcal{M}$ for interpreting terms, of an interpretation of function symbols $\mathcal{F}_\mathcal{M}(f) : \mathcal{D}^{a_f} \to \mathcal{D}$ and of an interpretation of atoms $\mathcal{P}_\mathcal{M}(\dot{P}) \subset \mathcal{D}^{a_{\dot{P}}}$ (for a_f, $a_{\dot{P}}$ the arity of f, $\dot{P}$ resp.).

Truth is defined by

$$
\begin{array}{lcl}
[\![x]\!]^\sigma_\mathcal{M} & \triangleq & \sigma(x) \\
[\![f t_1 \ldots t_{a_f}]\!]^\sigma_\mathcal{M} & \triangleq & \mathcal{F}_\mathcal{M}(f)([\![t_1]\!]^\sigma_\mathcal{M}, \ldots, [\![t_{a_f}]\!]^\sigma_\mathcal{M}) \\
\\
[\![\dot{P}(t_1, \ldots, t_{a_{\dot{P}}})]\!]'^\sigma_\mathcal{M} & \triangleq & \mathcal{P}_\mathcal{M}(\dot{P})([\![t_1]\!]^\sigma_\mathcal{M}, \ldots, [\![t_{a_{\dot{P}}}]\!]^\sigma_\mathcal{M}) \\
[\![\dot{\bot}]\!]'^\sigma_\mathcal{M} & \triangleq & \mathcal{P}_\mathcal{M}(\dot{\bot}) \\
[\![F \dot{\to} G]\!]'^\sigma_\mathcal{M} & \triangleq & [\![F]\!]'^\sigma_\mathcal{M} \Rightarrow\ [\![G]\!]'^\sigma_\mathcal{M} \\
[\![\dot{\forall} x\, F]\!]'^\sigma_\mathcal{M} & \triangleq & \forall t \in \mathcal{M}_D\ [\![F]\!]'^{\sigma[x \leftarrow t]}_\mathcal{M}
\end{array}
$$

Completeness w.r.t Tarskian models

Let $\mathcal{C}lassic$ be the theory containing $\dot{\neg}\dot{\neg}F\dot{\rightarrow}F$ for all formulas F (atoms are enough).

We define $\vdash_C F$ to be $\mathcal{C}lassic \vdash_M F$ in minimal logic.

A Tarskian model $\mathcal{M}$ for classical logic is a Tarskian model which satisfies $[\![\mathcal{C}lassic]\!]'_{\mathcal{M}}$ (in a classical meta-language, all Tarskian models are classical, but not in an intuitionistic meta-language).

The statement of completeness w.r.t Tarskian models for classical logic is:

$$[\forall \mathcal{M}\, \forall \sigma\, ([\![\mathcal{C}lassic]\!]'^{\sigma}_{\mathcal{M}} \Rightarrow [\![F]\!]'^{\sigma}_{\mathcal{M}})] \Rightarrow \mathcal{C}lassic \vdash_M F$$

The usual proof is by contradiction, building a saturated counter-model by enumeration of the formulas.

The proof with effects we shall consider actually works for arbitrary theories, so that we shall consider instead the following statement:

$$(\forall \mathcal{M}\, \forall \sigma\, [\![F]\!]'^{\sigma}_{\mathcal{M}}) \Rightarrow \vdash_M F$$

Completeness w.r.t. Kripke models

Kripke models

A Kripke model $\mathcal{K}$ is an increasing family of Tarskian models indexed over a set of worlds $\mathcal{W}_\mathcal{K}$ ordered by $\geq_\mathcal{K}$. In the absence of $\vee$ and $\exists$, it is enough to take $\mathcal{D}_\mathcal{K}$ constant.

Truth relatively to $\mathcal{K}$ at world w is defined by:

$$\begin{array}{lcl}
[\![x]\!]^\sigma_\mathcal{K} & \triangleq & \sigma(x) \\
[\![f t_1 \ldots t_{a_f}]\!]^\sigma_\mathcal{K} & \triangleq & \mathcal{F}_\mathcal{K}(f)([\![t_1]\!]^\sigma_\mathcal{K}, \ldots, [\![t_{a_f}]\!]^\sigma_\mathcal{K}) \\
 & & \\
w \Vdash^\sigma_\mathcal{K} \dot{P}(t_1 \ldots t_{a_{\dot{P}}}) & \triangleq & \mathcal{P}_\mathcal{K}(\dot{P})_w([\![t_1]\!]^\sigma_\mathcal{K}, \ldots, [\![t_{a_{\dot{P}}}]\!]^\sigma_\mathcal{K}) \\
w \Vdash^\sigma_\mathcal{K} \dot{\bot} & \triangleq & \mathcal{P}_\mathcal{K}(\dot{\bot})_w \\
w \Vdash^\sigma_\mathcal{K} F \dot{\rightarrow} G & \triangleq & \forall w' \geq_\mathcal{K} w\, (w' \Vdash^\sigma_\mathcal{K} F \Rightarrow w' \Vdash^\sigma_\mathcal{K} G) \\
w \Vdash^\sigma_\mathcal{K} \forall x\, F & \triangleq & \forall t \in \mathcal{K}_D\; w \Vdash^{\sigma[x \leftarrow t]}_\mathcal{K} F
\end{array}$$

The statement of completeness w.r.t. Kripke models is:

$$(\forall \mathcal{K}\, \forall \sigma\, \forall w \in \mathcal{W}_\mathcal{K}\; w \Vdash^\sigma_\mathcal{K} F) \Rightarrow \vdash_M F$$

Completeness w.r.t Kripke models

The "standard" proof works by building the canonical model $\mathcal{K}_0$ defined by taking $\mathcal{W}_{\mathcal{K}_0}$ to be the typing contexts ordered by inclusion, $\mathcal{D}_{\mathcal{K}_0}$ to be the terms, $\mathcal{K}_{\mathcal{F}}(f)$ to be the syntactic application of f, and $\mathcal{K}_{\mathcal{P}}(\dot{P})(\Gamma)(t_1, ..., t_{a_{\dot{P}}})$ to be $\Gamma \vdash_M \dot{P}(t_1, ..., t_{a_{\dot{P}}})$

The main lemma proves $\Gamma \vdash_M F \iff \Gamma \Vdash_{\mathcal{K}_0} F$ by induction on F

$$\begin{array}{lcll}
\uparrow^{\Gamma}_{F} & \Gamma \vdash_M F & \longrightarrow & \Gamma \Vdash_{\mathcal{K}_0} F \\
\uparrow^{\Gamma}_{\dot{P}(\vec{t})} & p & \triangleq & p \\
\uparrow^{\Gamma}_{F \dot{\to} G} & p & \triangleq & \Gamma' \mapsto h \mapsto m \mapsto \uparrow^{\Gamma'}_{G} \dot{\mathtt{App}}^{\Gamma',F,G}_{\Rightarrow}(\dot{\mathtt{weak}}^{\Gamma'}_{\Gamma,F}(h,p), \downarrow^{\Gamma'}_{F} m) \\
\uparrow^{\Gamma}_{\dot{\forall} x\, F} & p & \triangleq & t \mapsto \uparrow^{\Gamma}_{F[t/x]} \dot{\mathtt{App}}^{\Gamma,x,F}_{\forall}(p,t)
\end{array}$$

$$\begin{array}{lcll}
\downarrow^{\Gamma}_{F} & \Gamma \Vdash_{\mathcal{K}_0} F & \longrightarrow & \Gamma \vdash_M F \\
\downarrow^{\Gamma}_{\dot{P}(\vec{t})} & m & \triangleq & m \\
\downarrow^{\Gamma}_{F \dot{\to} G} & m & \triangleq & \dot{\mathtt{Abs}}^{\Gamma,F,G}_{\Rightarrow}(\downarrow^{\Gamma,F}_{G} (m\,(\Gamma,F)\, r^{\Gamma}_{F}\, (\uparrow^{\Gamma,F}_{F} \dot{\mathtt{Ax}}^{\Gamma_1,F,,\Gamma}(b_F)))) \\
\downarrow^{\Gamma}_{\dot{\forall} x\, F} & m & \triangleq & \dot{\mathtt{Abs}}^{\Gamma,x,F}_{\forall}(\dot{y}, \downarrow^{\Gamma}_{F[z/x]} (m\,\dot{y})) \qquad \dot{y} \text{ fresh in } \Gamma
\end{array}$$

And finally:

$$\mathtt{compl} \triangleq v \mapsto \downarrow^{\epsilon}_{A} (v\, \mathcal{K}_0\, \emptyset\, \epsilon) \;:\; (\forall \mathcal{K}\, \forall \sigma\, \forall w \in \mathcal{W}_{\mathcal{K}}\, w \Vdash^{\sigma}_{\mathcal{K}} F) \Rightarrow \vdash_M F$$

Completeness w.r.t. Kripke models in direct-style

Kripke forcing translation for second-order arithmetic

We consider a second-order arithmetic multi-sorted over first-order datatypes such as $\mathbb{N}$, lists, formulas, etc., and with primitive recursive atoms written $P(t_1, ..., t_{a_P})$.

$$A, B \triangleq X(t_1, ..., t_{a_X}) \mid P(t_1, ..., t_{a_P}) \mid A \wedge B \mid A \Rightarrow B \mid \forall x\, A \mid \forall X\, A$$

Let $\geq$ be a preorder. We extend Kripke forcing to second order quantification.

$$\begin{array}{lcl}
w \vDash X(t_1, ..., t_{a_X}) & \triangleq & X(w, t_1, ..., t_{a_X}) \\
w \vDash P(t_1, ..., t_{a_P}) & \triangleq & P(t_1, ..., t_{a_P}) \\
w \vDash A \wedge B & \triangleq & (w \vDash A) \wedge (w \vDash B) \\
w \vDash A \Rightarrow B & \triangleq & \forall w' \geq w\, [(w' \vDash A) \Rightarrow (w' \vDash B)] \\
w \vDash \forall x\, A & \triangleq & \forall x\, w \vDash A \\
w \vDash \forall X\, A & \triangleq & \forall X\, (mon(X) \Rightarrow w \vDash A)
\end{array}$$

where $mon(X) \triangleq \forall w\, \forall w' \geq w\, (X(w, t_1, ..., t_{a_X}) \Rightarrow X(w', t_1, ..., t_{a_X}))$

Relating completeness w.r.t Tarskian models to completeness w.r.t. Kripke models

We get a stronger statement of completeness by considering completeness w.r.t Kripke models by specifically instantiating $\mathcal{W}_\mathcal{K}$ to be the typing contexts and $\geq$ to be inclusion of contexts.

$$(\forall(\mathcal{D}_\mathcal{K}, \mathcal{F}_\mathcal{K}, \mathcal{P}_\mathcal{K})\, \forall\sigma\, [\epsilon \Vdash^\sigma_{(\mathcal{W}_\mathcal{K}, \mathcal{D}_\mathcal{K}, \mathcal{K}_\mathcal{F}, \mathcal{P}_\mathcal{K})} F]) \Rightarrow \vdash_M F$$

Now, applying forcing shows that

$$\epsilon \Vdash_x (\forall(\mathcal{D}_\mathcal{M}, \mathcal{F}_\mathcal{M}, \mathcal{P}_\mathcal{M})\forall\sigma \vDash_{(\mathcal{D}_\mathcal{M}, \mathcal{F}_\mathcal{M}, \mathcal{P}_\mathcal{M})} F)$$

is equivalent to

$$\forall(\mathcal{D}_\mathcal{K}, \mathcal{F}_\mathcal{K}, \mathcal{P}_\mathcal{K})\, \forall\sigma\, (\epsilon \Vdash_{(\mathcal{W}_\mathcal{K}, \mathcal{D}_\mathcal{K}, \mathcal{K}_\mathcal{F}, \mathcal{P}_\mathcal{K})} F)$$

and hence that *forcing over the statement of completeness w.r.t. Tarskian models is equivalent to the instantiation of the set of worlds to typing contexts of completeness w.r.t. Kripke models*

Excerpt of our meta-language with effects

$$\frac{\Gamma \vdash p : A(y) \qquad y \text{ fresh in } \Gamma}{\Gamma \vdash \lambda y.p : \forall y\, A(y)}\ \forall_I$$

$$\frac{\Gamma \vdash p : \forall x\, A(x) \qquad t \text{ updatable-variable-free or } t \text{ an updatable variable and } A(x) \text{ of type } 1}{\Gamma \vdash pt : A(t)}\ \forall_E$$

$$\frac{\Gamma \vdash p : A(X) \qquad X \text{ fresh in } \Gamma}{\Gamma \vdash p : \forall X\, A(X)}\ \forall_I^2 \qquad \frac{\Gamma \vdash p : \forall X\, A(X) \qquad \Gamma \vdash q : mon_\Gamma\, B(\vec{y})}{\Gamma \vdash p : A(X)[B(\vec{y})/X(\vec{y})]}\ \forall_E^2$$

$$\frac{\Gamma, [b : x \geq t] \vdash q : T(x) \qquad \Gamma \vdash r : \mathit{refl} \geq \qquad \Gamma \vdash s : \mathit{trans} \geq \qquad x \text{ fresh in } \Gamma \text{ and } T(t)}{\Gamma \vdash \mathtt{set}\, x := t\, \mathtt{as}\, b/_{(r,s)}\, \mathtt{in}\, q : T(t)}\ \text{SETEFF}$$

$$\frac{\Gamma, [b : x \geq t(x')] \vdash q : T(x) \quad \Gamma \vdash r : t(x') \geq x' \quad [x \geq u] \in \Gamma \text{ for some } u \quad x' \text{ fresh in } \Gamma}{\Gamma \vdash \mathtt{update}\, x {:=} t(x)\, \mathtt{of}\, x'\, \mathtt{as}\, b\, \mathtt{by}\, r\, \mathtt{in}\, q : T(t(x))}\ \text{UPDATE}$$

where C of type 1 means in the grammar $C ::= P(t_1, ..., t_{a_P}) \mid P(t_1, ..., t_{a_P}) \Rightarrow C \mid \forall x\, C$ and $mon_\Gamma\, B$ means B monotonic for all updatable variables in Γ

The completeness proof in direct-style

In direct style, $\mathcal{K}_0$ is the model $\mathcal{M}_0$ defined by $\mathcal{P}_{\mathcal{M}}(\dot{P})(t_1, ..., t_{a_{\dot{P}}}) \triangleq \Gamma \vdash \dot{P}(t_1, ..., t_{a_{\dot{P}}})$ for Γ a given updatable variable

$$
\begin{array}{llcl}
\uparrow_F & \Gamma \vdash_M F & \longrightarrow & [\![F]\!]'_{\mathcal{M}_0} \\
\uparrow_{P(\vec{t})} & g & \triangleq & g \\
\uparrow_{F \dot{\to} G} & g & \triangleq & m \mapsto \uparrow_G \mathtt{A\dot{p}p}_{\Rightarrow}^{\Gamma,F,G}(g, \downarrow_F m) \\
\uparrow_{\dot{\forall} x\, F} & g & \triangleq & t \mapsto \uparrow_{F[t/x]} \mathtt{A\dot{p}p}_{\forall}^{\Gamma,x,F}(g, t)
\end{array}
$$

$$
\begin{array}{llcl}
\downarrow_F & [\![F]\!]'_{\mathcal{M}_0} & \longrightarrow & \Gamma \vdash_M F \\
\downarrow_{P(\vec{t})} & m & \triangleq & m \\
\downarrow_{F \dot{\to} G} & m & \triangleq & \mathtt{A\dot{b}s}_{\Rightarrow}^{\Gamma,F,G}(\mathtt{update}\, \Gamma := (\Gamma, F)\, \mathtt{of}\, \Gamma_1\, \mathtt{as}\, b_F\, \mathtt{by}\, r_F^{\Gamma}\, \mathtt{in} \downarrow_G (m\, (\uparrow_F \mathtt{A\dot{x}}^{\Gamma_1,F,,\Gamma}(b_F)))) \\
\downarrow_{\dot{\forall} x\, F} & m & \triangleq & \mathtt{A\dot{b}s}_{\forall}^{\Gamma,x,F}(\dot{y}, \downarrow_{F[z/x]} (m\, \dot{y}))
\end{array}
$$

$$
\mathtt{compl} \triangleq v \mapsto \mathtt{set}\, \Gamma := \epsilon\, \mathtt{as}\, b/_{(r,s)}\, \mathtt{in} \downarrow_F^{\epsilon} (v\, \mathcal{M}_0\, \emptyset)
$$

Obviously, the resulting proof in the object language is a reification of the proof of validity as in Normalisation-by-Evaluation / semantic normalisation [C. Coquand 93, Danvy 96, Altenkirch-Hofmann 96, Okada 99, ...]

'e.

Status of the meta-language with update effect

- A certain degree of freedom in the design
  - Basic version using only Kripke forcing is inconsistent with classical logic
  - Local use of classical reasoning providing Markov's principle and Double Negation Shift are possible using Ilik's variant of Kripke forcing
  - A variant consistent with classical logic using Cohen forcing (but then completeness of intuitionistic logic w.r.t. Tarskian semantics not any more provable)
- Justification of the different variants by translation within intuitionistic logic
- Can be equipped with a reduction semantics (derived from the forcing interpretation)