Games, abstract machines and sequent calculus

Plan

1. *Sequent calculus*
2. *Proofs as strategies*
   - Coquand's reading of Gentzen and Novikoff
   - Lorenzen's E and D-dialogues
   - Girard's designs
3. *Cut-elimination as interaction*
   - Coquand's debate
   - Interaction and abstract machines

Cut-free classical sequent calculus

$$\frac{}{\Gamma, A \vdash \Delta, A}\ Ax \qquad\qquad \frac{\Gamma, A \vdash \Delta, A \to B, B}{\Gamma \vdash \Delta, A \to B} \to_R$$

$$\frac{\Gamma, A \to B \vdash \Delta, A \qquad \Gamma, A \to B, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta} \to_L$$

- subformula property

- admissibility of cut rule

- intuitionistic sequent calculus = at most one formula on the right

- minimal sequent calculus = exactly one formula on the right

Cut-free classical sequent calculus

$$\frac{\Gamma \vdash \Delta, A \wedge B, A \qquad \Gamma \vdash \Delta, A \wedge B, B}{\Gamma \vdash \Delta, A \wedge B} \wedge_R$$

$$\frac{\Gamma, A \wedge B, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge_L^1 \qquad \frac{\Gamma, A \wedge B, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge_L^2$$

Remark: No implication + classical logic + dual connectives: all formulas can be put on the right-hand side.

Coquand's game interpretation of Gentzen and Novikoff notion of "constructive" truth

- Gentzen's concept of "statability of a reduction rule for a sequent" serves as "the formal replacement of the informal concept of truth", providing us with a "finistic interpretation of propositions" and taking place of their "actualist interpretation" [1936]

Concretely: statability of a reduction rule means that a rule of cut-free ω-logic can be applied to refine the sequent

- Novikoff's notion of "regular" (or provable) formula

Gentzen's statability of a reduction rule (simplified)

$$A \ ::= \ P(x_1, ..., x_n) \mid \exists x.A \mid \forall x.A \qquad (P \text{ decidable})$$

$$\frac{P(n_1, ..., n_p) = 1}{\Gamma, P(n_1, ..., n_p) \text{ true}} \qquad \frac{\Gamma, \exists x.A(x), A(m) \text{ true}}{\Gamma, \exists x.A(x) \text{ true}}$$

$$\frac{\Gamma, A(0) \text{ true} \qquad \ldots \qquad \Gamma, A(n) \text{ true} \qquad \ldots}{\Gamma, \forall x.A(x) \text{ true}}$$

Fact: $\perp$ not true

Proposition: provable in arithmetics $\Rightarrow$ statability of reduction rules

Proof: admissibility of cut-elimination and therefore of induction

Theorem: consistency of arithmetics [Gentzen]

Coquand's reading of proofs as strategies

$$\frac{\Gamma, \exists x \forall y.A(x,y), A(m,0) \qquad \ldots \qquad \Gamma, \exists x \forall y.A(x,y), A(m,n) \qquad \ldots}{\Gamma, \exists x \forall y.A(x,y)}$$

Proof	=	strategy for Player
Proof step	=	move for Player
Choice of premise	=	move for Opponent
Well-foundedness of proof	=	Opponent forced to stop = Player wins

Two informations in a Player move (Curien's terminology):

pointer interaction = choice of $\exists x \forall y.A(x,y)$ in Γ

tree interaction = choice of m

Lorenzen's approach

A game between Player and Opponent (by Felscher: P and Q)

Play = sequence of alternating moves by Player and Opponent

Move = attack or answer

Formula $A \to B$ can be attacked by move $[_A$ and the only answer is $]_B$

Formula $A \wedge B$ can be attacked by move $[_1$ and the only answer is $]_A$

Formula $A \wedge B$ can be attacked by move $[_2$ and the only answer is $]_B$

Player wins = Opponent cannot move.

Proposition: A provable iff there is a winning strategy for Player

Lorenz and Lorenzen's approach

Two kinds of games:

E-dialogues	$=$	no pointer interaction for Opponent
	$=$	Coquand's approach
	$\Rightarrow$	P-strategies are isomorphic to (well-suited) proofs
D-dialogues	$=$	(almost) symmetric rôles for Player and Opponent
	$=$	Hyland-Ong style of game without Opponent repetitions

According to Felscher [1984], D-dialogues are due to Lorenz [1961] and E-dialogues to Lorenzen and Schwemmer [1973]

Lorenzen E-dialogues = LJQ

P-strategy	=	(cut-free) proof of LJQ [PhD]
	=	(trace of) innocent strategy in Honda-Yoshida CBV game

$$\frac{}{\Gamma, A \vdash A}\ ]^P_A \qquad\qquad \frac{\overline{\Gamma \vdash A} \qquad \begin{matrix}\vdots\ ]^O_B \\ \Gamma, B \vdash C\end{matrix}}{\Gamma, A \to B \vdash C}\ [^P_A$$

$$\frac{\begin{matrix}\vdots\ [^O_A \\ \Gamma, A \vdash B\end{matrix}}{\Gamma \vdash A \to B}\ ]^P_{A \to B} \qquad\qquad \frac{\begin{matrix}\vdots\ [^O_{A_1} \\ \Gamma, A_1 \vdash A_2\end{matrix} \qquad \begin{matrix}\vdots\ ]^O_B \\ \Gamma, B \vdash C\end{matrix}}{\Gamma, (A_1 \to A_2) \to B \vdash C}\ [^P_{A_1 \to A_2}$$

Remark: introduction rules: left $= [^P$ and right $=]^P$

Lorenzen E-dialogues = LJQ

$$
\dfrac{\begin{array}{c}\vdots\;[^{O}_{1}\\ \Gamma \vdash A\end{array} \qquad \begin{array}{c}\vdots\;[^{O}_{1}\\ \Gamma \vdash B\end{array}}{\Gamma \vdash A \wedge B}\;]^{P}_{A\wedge B}
\qquad\qquad
\dfrac{\begin{array}{c}\vdots\;[^{O}_{\wedge 1}\\ \Gamma \vdash A_1\end{array} \qquad \begin{array}{c}\vdots\;[^{O}_{\wedge 2}\\ \Gamma \vdash A_2\end{array} \qquad \begin{array}{c}\vdots\;]^{O}_{B}\\ \Gamma, B \vdash C\end{array}}{\Gamma, (A_1 \wedge A_2) \to B \vdash C}\;[^{P}_{A_1\wedge A_2}
$$

$$
\dfrac{\begin{array}{c}\vdots\;]^{O}_{A}\\ \Gamma \vdash A\end{array}}{\Gamma, A \wedge B \vdash}\;[^{P}_{\wedge 1}
\qquad\qquad
\dfrac{\begin{array}{c}\vdots\;]^{O}_{B}\\ \Gamma \vdash B\end{array}}{\Gamma, A \wedge B \vdash}\;[^{P}_{\wedge 2}
$$

Remark: interpretation of $A \wedge B$ differs in Honda-Yoshida CBV games

Anedocte: LJQ is the intuitionistic version of Danos-Joinet-Schellinx LKQ; this calculus has a dual called LKT whose intuitionistic restriction is LJT; not to be confused with Roy Dyckhoff LJT calculus which is similar to above LJQ

The symmetrisation of E-dialogues

Rules of E-dialogues are not symmetric: P- and O-strategies cannot interact

Rules of D-dialogues rules are almost symmetric: P- and O-strategies can interact if Opponent doesn't play 2 times the same move

Spread E-dialogues = D-dialogues + Opponent allowed to repeat: P- and O-strategies can interact but infinite play may occur and Player is never able to win (unless adding a fairness condition)

In spread E-dialogues : P- and O-strategies = innocent strategies [TLCA97]

Lorenzen's approach: the classical case

E and D-dialogues include a well-bracketing condition, typical to intuitionistic games (including HO, AJM, HY games)

C-dialogues $=$ D-dialogues without bracketing condition

$\Rightarrow$ A classically provable iff there is a winning P-strategy

E-dialogues and no bracketing $\Rightarrow$ P-strategy = proof of LKQ

Conjecture: The symmetrisation of P-strategies in an E-dialogue without bracketing should correspond to classical call-by-value Honda-Yoshida style games.

Proposition: The symmetrisation of proofs of classical PCF seen as P-strategies yields HO-style innocent strategies for classical PCF [TLCA97]

Girard's design

Designs (dessins/drawings) = partial proofs of this sequent calculus:

$$
\begin{array}{rcl}
P & ::= & \oplus_{I \in \mathcal{P}_f(\mathbb{N})} \otimes_{i \in I} N_i \\
N & ::= & \&_{I \in \mathcal{P}_f(\mathbb{N})} \parr_{i \in I} P_i
\end{array}
$$

$$
\frac{}{\Gamma}\ \mho \qquad\qquad \frac{}{\Gamma}\ \perp
$$

$$
\frac{\ldots \Gamma_i ; N_i \ldots}{\Gamma, \oplus_{I \in \mathcal{P}_f(\mathbb{N})} \otimes_{i \in I} N_i}
\qquad\qquad
\frac{\ldots (\Gamma, P_{i_1} \ldots P_{i_n})_{\{i_1 \ldots i_n\}} \ldots}{\Gamma ; \&_{\{i_1 \ldots i_n\} \in \mathcal{P}_f(\mathbb{N})} \parr_{i \in I} P_i}
$$

Similar to Coquand's strategies but handles linear logic connectives

Coquand's approach of normalisation-as-interaction

Coquand's approach handles strategies in E-dialogue P-strategies style. They don't compose immediately.

Coquand's debate: a protocol to let P-strategies in A and $\neg A$ interact.

Theorem: interaction between well-founded proofs terminates [Coquand]

Proof: relies on interaction sequence (the pure pointer interaction information)

Proposition: interaction mimics weak-head cut-elimination [PhD]

Interaction and abstract machines

Abstract machines implement "head linear reduction" (= "weak-head reduction")

Example: *stack-free Krivine Abstract Machine*

Reduction of $\lambda\overrightarrow{x_0}.(x_{0j}\,\overrightarrow{u})$ applied to $\overrightarrow{t}$

Initial state: $(x_{0j}\,\overrightarrow{u})[\overrightarrow{x_0} := \overrightarrow{t}[]]$

Transition: $(x_{ij}\,\overrightarrow{u})[e] \to (z\,\overrightarrow{t})[\overrightarrow{y} := \overrightarrow{u}[e]; e_i]$

where $e = \overrightarrow{x_1} := \overrightarrow{t_1}[e_1] \ldots \overrightarrow{x_n} := \overrightarrow{t_n}[e_n]]$ and $t_i = \lambda\overrightarrow{y}.(z\,\overrightarrow{t})$

Interaction and abstract machines

Example:

Evaluation of $(\lambda f, x.f(fx))(\lambda y.y)2$

$f(fx)[f := \lambda y.y; x := 2]$
$\to y[y := fx[f := \lambda y.y; x := 2]]$
$\to fx[f := \lambda y.y; x := 2]$
$\to y[y := x[f := \lambda y.y; x := 2]]$
$\to x[f := \lambda y.y; x := 2]$
$\to 2$

Interaction and abstract machines

Proposition: interaction mimics reduction in an abstract machine [Danos-Regnier, LICS96]

General pattern:

- x's (= P-moves) and y's (= O-moves) alternate
- indice into environment = pointer into the history of interaction

Interaction and abstract machines

The Pointer Abstract Machine [Danos-Regnier]

"PAM = Pascal language execution model (in Call-By-Name)"

(stack-based representation of environment)

THE END