Kripke Models for Classical Logic

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Abstract

We introduce a notion of Kripke model for classical logic for which we constructively prove soundness and cut-free completeness. We discuss the novelty of the notion and its potential applications.

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1. Introduction

Kripke models have been introduced as means of giving semantics to modal logics and were later used to give semantics for intuitionistic logic as well.[22, 23] The purpose of the present paper is to show that Kripke models can also be used as semantics for *classical* logic. Of course, Kripke semantics can be indirectly assigned to classical logic by means of some appropriate double-negation translation, as in [3], but our goal here is to provide a *direct* presentation of a notion of Kripke semantics for classical logic.

We will use the $\mathrm{LK}_{\mu\tilde{\mu}}$ sequent calculus of [8] to represent proofs, but the conclusions given apply to any complete formal system for classical logic. There are at least two reasons for choosing $\mathrm{LK}_{\mu\tilde{\mu}}$: first, it is a typing system for a calculus very close to λ-calculus and we are ultimately interested in the computational content of classical logic; second, the symmetry of left/right distinguished formulae of $\mathrm{LK}_{\mu\tilde{\mu}}$ allows to give two dual notions of models, of which only one needs to be, and is, presented in this paper, while the other can be derived by analogy.

This paper is organised as follows. Section 2 introduces the notion of classical Kripke model, based on two modifications to the traditional notion, and discusses the relationship between the traditional and our notion. Section 3 introduces the sequent

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calculus $LK_{\mu\tilde{\mu}}$ and gives a soundness theorem for it. Section 4 proves a completeness theorem for a universal model constructed from the deduction system itself. Section 5 is the concluding section which discusses related and future work.

We use the standard inductive definition of first-order formulae for the connectives $\{\top, \bot, \wedge, \vee, \rightarrow, \exists, \forall\}$. The language has infinitely many constants. A sentence is a formula where all variables are bound by quantifiers. An atomic formula is one which is not built up from logical connectives, i.e. it is one built up of a predicate symbol. The shorthand $\neg A$ stands for $A \rightarrow \bot$.

All statements and proofs are constructive.

2. Classical Kripke Models

Kripke models can be considered as the "most classical" of all the semantics for intuitionistic logic, for two reasons: first, each of the 'possible worlds' that define a Kripke model is a classical world in itself (where either an atom or its negation are true); second, it is the single of the semantics for intuitionistic logic which has only a classical proof of completeness, when disjunction and existential quantification are considered.[1]

In the last two decades, the Curry-Howard correspondence between intuitionistic proof systems and typed lambda-calculi has been extended to classical proof systems [17, 29, 8]. The idea for introducing direct-style Kripke models for classical logic came from their usefulness in providing normalisation-by-evaluation for intuitionistic proof systems [6, 7]. To account for a classical proof system we modify the traditional notion of Kripke model in the following two ways.

Not taking the forcing relation as primitive. We take as primitive the notion of "strong refutation", and define forcing in terms of it.[2] The forcing definition we get in this way partly coincides with the traditional definition of forcing, as explained in subsection 2.1.

Allowing certain nodes to validate absurdity. We allow certain possible worlds to be marked as "fallible", or "exploding". This approach has been taken for Kripke models in [35], for Beth models by Friedman [31] and is necessary in order to have a constructive proof of completeness, in the view of the meta-mathematical results from [21, 26, 27], which preclude constructive proofs[3] of completeness in case one wants to retain that absurdity must never be valid in a possible world[4].

Definition 1. *A* classical Kripke model *is given by a quintuple* $(K, \leq, D, \Vdash_{\overline{s}}, \Vdash_{\bot})$*, K inhabited, such that*

[1] There is an intuitionistic proof in [35], but it makes use of the fan theorem which is not universally recognised as constructive.

[2] For an alternative, see the discussion on dual models in Section 5.

[3] Strictly speaking, the cited results show that having a constructive proof of completeness implies having a proof of Markov's Principle.

[4] Extending the class of Boolean models with inconsistent models is also the key to the constructive proof of the classical completeness theorem in [24]. For an analysis of that result, see [4].

- $(K, \leq)$ *is a poset of "possible worlds";*
- D *is the "domain function" assigning sets to the elements of* K *such that*

$$\forall w, w' \in K, (w \leq w' \Rightarrow D(w) \subseteq D(w'))$$

  i.e., D *is monotone;*

  Let the language be extended with constant symbols for each element of $\mathcal{D} := \cup\{D(w) : w \in K\}$.

- $(-) : (-) \Vdash_s$ *is a binary relation of "strong refutation" between worlds and atomic sentences in the extended language such that*
  - $w : X(d_1, ..., d_n) \Vdash_s \quad \Rightarrow \quad d_i \in D(w)$ *for each* $i \in \{1, ..., n\}$,
  - *(Monotonicity)* $w : X(d_1, ..., d_n) \Vdash_s \;\&\; w \leq w' \;\Rightarrow\; w' : X(d_1, ..., d_n) \Vdash_s$,
- $(-) \Vdash_\perp$ *is a unary relation on worlds labelling a world as "exploding", which is also monotone in the above sense.*

The strong refutation relation is extended from atomic to composite sentences inductively and by mutually defining the relations of *forcing* and (non-strong) *refutation*.

Definition 2. *The relation* $(-) : (-) \Vdash_s$ *of* strong refutation is extended to *the relation between worlds* w *and* composite sentences A *in the extended language with constants in* $D(w)$*, inductively, together with the two new relations:*

- *A sentence* A *is* forced *in the world* w *(notation* $w :\Vdash A$*) if any world* $w' \geq w$*, which strongly refutes* A*, is exploding;*
- *A sentence* A *is* refuted *in the world* w *(notation* $w : A \Vdash$*) if any world* $w' \geq w$*, which forces* A*, is exploding;*
- $w : A \wedge B \Vdash_s$ *if* $w : A \Vdash$ *or* $w : B \Vdash$*;*
- $w : A \vee B \Vdash_s$ *if* $w : A \Vdash$ *and* $w : B \Vdash$*;*
- $w : A \to B \Vdash_s$ *if* $w :\Vdash A$ *and* $w : B \Vdash$*;*
- $w : \forall x.A(x) \Vdash_s$ *if* $w : A(d) \Vdash$ *for some* $d \in D(w)$*;*
- $w : \exists x.A(x) \Vdash_s$ *if, for any* $w' \geq w$ *and* $d \in D(w')$, $w' : A(d) \Vdash$*;*
- $\perp$ *is always strongly refuted;*
- $\top$ *is never strongly refuted.*

The notions of forcing and refutation can be somewhat understood as the classical notions of being true and being false. However, a statement of form $P \Rightarrow w \Vdash_\perp$ should not be thought of as negation of P at the meta-level, because in the concrete model we provide in section 4, $w \Vdash_\perp$ is always an inhabited set. In other words, we never use *ex falso quodlibet* at the meta-level to handle exploding nodes.

The notion of strong refutation is more informative than the notion of (non-strong) refutation, not only because the former implies the latter, but also because, for example, having $w : A \wedge B \Vdash_{\mathfrak{s}}$ tells use which one of A, B is refuted, while $w : A \wedge B \Vdash$ does not.

A more detailed characterisation of the notions is given in the rest of this section.

Lemma 3. *Strong refutation, forcing and refutation are monotone in any classical Kripke model.*

Proof. The monotonicity of strong refutation can be proved by induction on the formula in question, while that of forcing and refutation is obviously true. □

Lemma 4. *Strong refutation implies refutation: In any world w and for any sentence A, $w : A \Vdash_{\mathfrak{s}}$ implies $w : A \Vdash$.*

Proof. Suppose $w : A \Vdash_{\mathfrak{s}}$, $w' \geq w$ and $w' :\Vdash A$. Then w' is exploding because $w' : A \Vdash_{\mathfrak{s}}$ by monotonicity. Since w' was arbitrary, $w : A \Vdash$. □

2.1. Relation to Traditional Forcing and Further Properties

It is natural to ask what is the relationship between traditional intuitionistic forcing[31] and our forcing whose definition relies on a more primitive notion. Lemmas 5 and 8 give that the two notions coincide on the fragment of formulae constructed by $\{\rightarrow, \wedge, \forall, \top\}$

Lemma 5. *The following statements hold.*

$$
\begin{aligned}
w :\Vdash A \rightarrow B &\iff \textit{for all } w' \geq w, w' :\Vdash A \Rightarrow w' :\Vdash B && (1)\\
w :\Vdash A \wedge B &\iff w :\Vdash A \textit{ and } w :\Vdash B && (2)\\
w :\Vdash \forall x.A(x) &\iff \textit{for all } w' \geq w \textit{ and } d \in D(w'), w' :\Vdash A(d) && (3)\\
w :\Vdash A \vee B &\Longleftarrow w :\Vdash A \textit{ or } w :\Vdash B && (4)\\
w :\Vdash \exists x.A(x) &\Longleftarrow \textit{for some } d \in D(w), w :\Vdash A(d) && (5)
\end{aligned}
$$

Proof. Lemma 3 and Lemma 4 are used implicitly in the following proof.

(1) Left-to-right: Suppose $w' \geq w$ and $w' :\Vdash A$. To show $w' :\Vdash B$ we let $w'' \geq w'$ and $w'' : B \Vdash_{\mathfrak{s}}$ and have to show that w'' is exploding. Since then $w'' : A \rightarrow B \Vdash_{\mathfrak{s}}$ holds by monotonicity and Lemma 4, the claim follows from the definition of $w :\Vdash A \rightarrow B$.

Right-to-left: Suppose $w' \geq w$ and $w' : A \rightarrow B \Vdash_{\mathfrak{s}}$, i.e., $w' :\Vdash A$ and $w' : B \Vdash$. We have to show w' is exploding. But, this is immediate, since $w' :\Vdash B$ by assumption.

(2) Left-to-right: Suppose $w' \geq w$ and $w' : A \Vdash_{\mathfrak{s}}$. Then $w' : A \Vdash$, and so $w' : A \wedge B \Vdash_{\mathfrak{s}}$. This implies that w' is exploding, that is, $w :\Vdash A$. Similarly, we can show $w :\Vdash B$.

Right-to-left: Suppose $w' \geq w$ and $w' : A \wedge B \Vdash_{\mathfrak{s}}$. Therefore we have $w' : A \Vdash$ or $w' : B \Vdash$. Each case leads to $w' :\Vdash_{\perp}$ since $w' :\Vdash A$ and $w' :\Vdash B$ by monotonicity.

(3) Left-to-right: Suppose $w'' \geq w' \geq w$, $d \in D(w')$, and $w'' : A(d) \Vdash_{\mathfrak{s}}$. Then $w'' : \forall x.A(x) \Vdash_{\mathfrak{s}}$, so w'' is exploding.

Right-to-left: Suppose $w' \geq w$ and $w' : \forall x.A(x) \Vdash_{\mathfrak{s}}$, i.e., $w' : A(d) \Vdash$ for some $d \in D(w')$. So w' is exploding by assumption.

The rest of the cases are obvious. □

Note, however, that although the definitions of our and intuitionistic forcing match on the fragment $\{\rightarrow, \wedge, \forall, \top\}$, that does not mean that a formula in that fragment is forced in our sense if and only if it is forced in the intuitionistic sense. The law of Peirce $((A \rightarrow B) \rightarrow A) \rightarrow A$ is one counterexample to that, it is classically but not intuitionistically forced.

Remark 6. *The following do not hold in general, even if reasoning classically.*

- $w :\Vdash A \vee B \Longrightarrow w :\Vdash A$ *or* $w :\Vdash B$.
- $w :\Vdash \exists x.A(x) \Longrightarrow$ *for some* $t \in D(w), w :\Vdash A(t)$.

The explanation is deferred to Remark 20.

Lemma 7. *Given a classical Kripke model $\mathcal{K}$, the following hold.*

1. $w : A \rightarrow B \Vdash$ *iff* $w : A \rightarrow B \Vdash_{\mathfrak{s}}$.
2. $w : A \vee B \Vdash$ *iff* $w : A \vee B \Vdash_{\mathfrak{s}}$.
3. $w : \exists x.A(x) \Vdash$ *iff* $w : \exists x.A(x) \Vdash_{\mathfrak{s}}$.
4. *If* $w : A \Vdash$ *or* $w : B \Vdash$*, then* $w : A \wedge B \Vdash$.
5. *If* $w : A(d) \Vdash$ *for some* $d \in D(w)$*, then* $w : \forall x.A(x) \Vdash$.

Proof. 1. Right-to-left is Lemma 4.
Left-to-right: Suppose $w' \geq w$ and $w' : A \Vdash_{\mathfrak{s}}$. In order to show that w' is exploding it suffices to show $w' :\Vdash A \rightarrow B$. For this assume $w'' \geq w'$ and $w'' : A \rightarrow B \Vdash_{\mathfrak{s}}$, i.e., $w'' :\Vdash A$ and $w'' : B \Vdash$. Then w'' is exploding since we have $w'' : A \Vdash_{\mathfrak{s}}$ by monotonicity. Similarly, we can show $w : B \Vdash$.
2. Right-to-left is Lemma 4.
Left-to-right: Suppose $w' \geq w$ and $w' :\Vdash A$. Then by Lemma 5, $w' :\Vdash A \vee B$ holds. So w' is exploding. That is $w : A \Vdash$. Similarly, $w : B \Vdash$ holds.
3. Right-to-left is Lemma 4.
Left-to-right: Suppose $w'' \geq w' \geq w$, $d \in D(w')$ and $w'' :\Vdash A(d)$. Then by Lemma 5, $w'' :\Vdash \exists x.A(x)$. So w'' is exploding since we have $w'' : \exists x.A(x) \Vdash$ by monotonicity.
4. Suppose w.l.o.g. $w : A \Vdash$, $w' \geq w$ and $w' :\Vdash A \wedge B$. Then by Lemma 5, $w' :\Vdash A$. So w' is exploding because we have $w' : A \Vdash$ by monotonicity.
5. Suppose $w' \geq w$ and $w' :\Vdash \forall x.A(x)$. Then by Lemma 5, $w' :\Vdash A(d)$. So w' is exploding because we have $w' : A(d) \Vdash$ by monotonicity.

□

We can also say that forcing of $\perp$ and $\top$ behaves like expected with respect to exploding nodes [35, 24]:

Lemma 8. 1. $w :\Vdash \top$ *and* $w : \bot \Vdash$.

2. *w is exploding iff $w :\Vdash \bot$.*
3. *w is exploding iff $w : \top \Vdash$.*

Proof. 1. Obvious.

2. Let w be an arbitrary world.

$$\begin{aligned} w :\Vdash \bot &\iff \forall (w' \geq w)\,(w' : \bot \Vdash_{\overline{s}} \Rightarrow w' :\Vdash_{\bot}) \\ &\iff \forall (w' \geq w)\,(w' :\Vdash_{\bot}) \iff w :\Vdash_{\bot} \end{aligned}$$

3. Similar.

□

We can use the previous lemmas to show that the forcing relation for classical logic behaves "classically" indeed:

Lemma 9. *The following hold in the classical Kripke semantics.*

1. $w :\Vdash A \iff w : \neg A \Vdash_{\overline{s}}$.
2. $w : A \Vdash \iff w :\Vdash \neg A$.
3. $w : \neg A \Vdash \iff w :\Vdash A$.
4. $w : \neg A \Vdash \iff w : \neg A \Vdash_{\overline{s}}$.
5. $w :\Vdash A \iff w :\Vdash \neg\neg A$.
6. $w : A \Vdash \iff w : \neg\neg A \Vdash$.
7. $w : \neg A \Vdash_{\overline{s}} \iff w :\Vdash \neg\neg A \Vdash \iff w :\Vdash A$.

Proof. 1. Obvious by definition because $w : \bot \Vdash$.

2. It follows from Lemma 5.
3. Obvious by Lemma 7 and the previous claims.
4. ∼ 7. Obvious from the previous claims.

□

Corollary 10. *In any classical Kripke model, the following holds.*

$$w : \neg A \Vdash_{\overline{s}} \iff w :\Vdash \neg\neg A \iff w :\Vdash A$$

We now consider the following double-negation translation $(\cdot)^*$, which is the one of Gödel-Gentzen[16, 15], except that atomic formulae, $\bot$ and $\top$ are not doubly negated:

$$\begin{aligned} X^* &:= X \quad (X \text{ is atomic}, \bot \text{ or } \top) \\ (A \wedge B)^* &:= A^* \wedge B^* \\ (A \rightarrow B)^* &:= A^* \rightarrow B^* \\ (\forall x.A)^* &:= \forall x.A^* \\ (A \vee B)^* &:= \neg(\neg A^* \wedge \neg B^*) \\ (\exists x.A)^* &:= \neg \forall x.\neg A^* \end{aligned}$$

Proposition 11. *Every classical Kripke model $\mathcal{C} = (K, \leq, D, \Vdash_{\overline{s}}, \Vdash_\perp)$ gives rise to an intuitionistic Kripke model with exploding worlds $\mathcal{I} = (K, \leq, D, \Vdash_i, \Vdash_\perp)$, which inherits all components of $\mathcal{C}$, except for $\Vdash_i$, which is defined for atomic formulae by non-strong forcing, i.e.*

$$w \Vdash_i X \text{ iff } w :\Vdash X$$

The translation $(\cdot)^*$ relates $\mathcal{C}$ and $\mathcal{I}$, that is, for any world w and any formula A we have

$$w \Vdash_i A^* \text{ iff } w :\Vdash A.$$

Proof. By induction on the complexity of A and by using (1)-(3) from Lemma 5 and (2) from Lemma 8. We detail only the induction case for $\vee$, which is the most involved one:

$$\begin{array}{rcl}
 & w \Vdash_i (A \vee B)^* & \iff \\
 & w \Vdash_i \neg(\neg A^* \wedge \neg B^*) & \iff \\
(\forall w' \geq w) & [w' \Vdash_i \neg A^*, w' \Vdash_i \neg B^* \Longrightarrow w' \Vdash_i \perp] & \iff \\
(\forall w' \geq w)[& (\forall w'' \geq w')[w'' \Vdash_i A^* \Longrightarrow w'' \Vdash_i \perp], & \\
 & (\forall w'' \geq w')[w'' \Vdash_i B^* \Longrightarrow w'' \Vdash_i \perp] & \\
 & \Longrightarrow w' \Vdash_i \perp] & \iff \\
(\forall w' \geq w)[& (\forall w'' \geq w')[w'' :\Vdash A \Longrightarrow w'' \Vdash_\perp], & \\
 & (\forall w'' \geq w')[w'' :\Vdash B \Longrightarrow w'' \Vdash_\perp] & \\
 & \Longrightarrow w' \Vdash_\perp] & \iff \\
(\forall w' \geq w) & [w' : A \Vdash, w' : B \Vdash \Longrightarrow w' \Vdash_\perp] & \iff \\
(\forall w' \geq w) & [w' : A \vee B \Vdash_{\overline{s}} \Longrightarrow w' \Vdash_\perp] & \iff \\
 & w :\Vdash A \vee B &
\end{array}$$

□

3. $\mathrm{LK}_{\mu\tilde{\mu}}$ and Soundness

To emphasise the symmetries of classical logic, we use a sequent calculus in the style of Gentzen's LK as proof system. We could have directly used LK or one of its variants with implicit structural rules, *à la* Kleene-Kanger. In practise, even though the current paper does not go into the details of the computational content of proofs, we rely here on $\mathrm{LK}_{\mu\tilde{\mu}}$ which has a simple symmetrical variant of λ-calculus as underlying language of proofs [8, 18][5].

$\mathrm{LK}_{\mu\tilde{\mu}}$ is presented on Table 1. It differs from LK in the following points:

- Sequents come with an explicitly distinguished formula on the right or on the left, or no distinguished formula at all, resulting in three kinds of sequents: "$\Gamma \vdash \Delta$", "$\Gamma | A \vdash \Delta$" and "$\Gamma \vdash A | \Delta$". Especially, the distinguished formula plays an "active" rôle in the rules.

[5]Note that even if not based on λ-calculus, there are calculi of proof-terms for LK too, see e.g. [32, 25, 34].

$\dfrac{}{\Gamma \vert A \vdash A, \Delta}\,(\mathrm{Ax}_L)$	$\dfrac{}{A, \Gamma \vdash A \vert \Delta}\,(\mathrm{Ax}_R)$
$\dfrac{\Gamma, A \vdash \Delta}{\Gamma \vert A \vdash \Delta}\,(\tilde{\mu})$	$\dfrac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vert \Delta}\,(\mu)$
$\dfrac{\Gamma \vdash A \vert \Delta \qquad \Gamma \vert B \vdash \Delta}{\Gamma \vert A \to B \vdash \Delta}\,(\to_L)$	$\dfrac{\Gamma, A \vdash B \vert \Delta}{\Gamma \vdash A \to B \vert \Delta}\,(\to_R)$
$\dfrac{\Gamma \vert A \vdash \Delta \qquad \Gamma \vert B \vdash \Delta}{\Gamma \vert A \vee B \vdash \Delta}\,(\vee_L)$	$\dfrac{\Gamma \vdash A \vert \Delta}{\Gamma \vdash A \vee B \vert \Delta}\,(\vee_R^1) \qquad \dfrac{\Gamma \vdash B \vert \Delta}{\Gamma \vdash A \vee B \vert \Delta}\,(\vee_R^2)$
$\dfrac{\Gamma \vert A \vdash \Delta}{\Gamma \vert A \wedge B \vdash \Delta}\,(\wedge_L^1) \qquad \dfrac{\Gamma \vert B \vdash \Delta}{\Gamma \vert A \wedge B \vdash \Delta}\,(\wedge_L^2)$	$\dfrac{\Gamma \vdash A \vert \Delta \qquad \Gamma \vdash B \vert \Delta}{\Gamma \vdash A \wedge B \vert \Delta}\,(\wedge_R)$
$\dfrac{\Gamma \vert A(x) \vdash \Delta \qquad x \text{ fresh}}{\Gamma \vert \exists x A(x) \vdash \Delta}\,(\exists_L)$	$\dfrac{\Gamma \vdash A(t) \vert \Delta}{\Gamma \vdash \exists x.A(x) \vert \Delta}\,(\exists_R)$
$\dfrac{\Gamma \vert A(t) \vdash \Delta}{\Gamma \vert \forall x.A(x) \vdash \Delta}\,(\forall_L)$	$\dfrac{\Gamma \vdash A(x) \vert \Delta \qquad x \text{ fresh}}{\Gamma \vdash \forall x A(x) \vert \Delta}\,(\forall_R)$
$\dfrac{}{\Gamma \vert \bot \vdash \Delta}\,(\bot_L)$	$\dfrac{}{\Gamma \vdash \top \vert \Delta}\,(\top_R)$
$\dfrac{\Gamma \vdash A \vert \Delta \qquad \Gamma \vert A \vdash \Delta}{\Gamma \vdash \Delta}\,(\mathrm{Cut})$	

Table 1: The sequent calculus $\mathrm{LK}_{\mu\tilde{\mu}}$

- Accordingly, the axiom rule splits into two variants (Ax_L) and (Ax_R) depending on whether the left active formula or the right active formula is distinguished. There are also two new rules, (μ) and $(\tilde{\mu})$, for making a formula active[6].
- There are no explicit contraction rules: contractions are derivable from a cut against an axiom as follows:
  - Left contraction:

$$\dfrac{\dfrac{}{\Gamma, A \vdash A \mid \Delta}\,(\mathrm{Ax}_R) \qquad \Gamma, A \mid A \vdash \Delta}{\Gamma, A \vdash \Delta}\,(\mathrm{Cut}) \qquad (Contr_L)$$

[6]Note that we have to define the contexts of formulae Γ and Δ as ordered sequences to get a non ambiguous interpretation of $\mathrm{LK}_{\mu\tilde{\mu}}$ as a typed λ-calculus. In this case, the notation A, Δ has to be understood as Δ_1, A, Δ_2 for Δ_1, Δ_2 a split of Δ.

– Right contraction:

$$\dfrac{\Gamma \vdash A \mid A, \Delta \qquad \dfrac{}{\Gamma \mid A \vdash A, \Delta}\,(\mathrm{Ax}_L)}{\Gamma \vdash A, \Delta}\,(\mathrm{Cut}) \qquad (Contr_R)$$

- Consequently, the notion of normal proof, or cut-freeness, is slightly different from the notion of cut-freeness in LK: a *normal proof* is a proof whose only cuts are of the form of a cut between an axiom and an introduction rule[7]. This is the notion that we refer to when below, very often, we say "cut-free" or "provable without a cut".

The correspondence between normal proofs of LK and normal proofs of $LK_{\mu\tilde{\mu}}$ is direct. If we present LK with weakening rules attached to the axiom rules *à la* Kleene's G_4 or Kanger's LC, we obtain an LK proof from an $LK_{\mu\tilde{\mu}}$ proof by erasing the bars serving to distinguish active formulae, and by removing the trivial inferences coming from the rules (μ) and ($\tilde{\mu}$). In the other way round, every introduction rule of LK can be derived in $LK_{\mu\tilde{\mu}}$ by applying the rules (μ) and ($\tilde{\mu}$) on the premises and a (possibly dummy) contraction (i.e. a cut against an axiom) on the conclusion of the rule. Similarly for the axiom rule (for which there are two possible derivations) and the cut rule. For more details we refer the reader to [8].

For a constant c, let $\Gamma_c(t), \Delta_c(t), A_c(t)$ be obtained from Γ, Δ, A by replacing each constant c with a term t.

Lemma 12 (Weakening)**.** *Suppose* $\Gamma \subseteq \Gamma'$ *and* $\Delta \subseteq \Delta'$.

- $\Gamma \vdash \Delta$ *implies* $\Gamma' \vdash \Delta'$.
- $\Gamma \vdash A \mid \Delta$ *implies* $\Gamma' \vdash A \mid \Delta'$.
- $\Gamma \mid A \vdash \Delta$ *implies* $\Gamma' \mid A \vdash \Delta'$.

Moreover, no further cuts in the derivations on the right-hand side are necessary.

Lemma 13. *Let c be a constant and y a variable which does not appear in* Γ, Δ, A.

- $\Gamma \vdash \Delta$ *implies* $\Gamma_c(y) \vdash \Delta_c(y)$.
- $\Gamma \vdash A \mid \Delta$ *implies* $\Gamma_c(y) \vdash A_c(y) \mid \Delta_c(y)$.
- $\Gamma \mid A \vdash \Delta$ *implies* $\Gamma_c(y) \mid A_c(y) \vdash \Delta_c(y)$.

Moreover, no further cuts in the derivations on the right-hand side are necessary.

The following lemma says that a fresh constant is as good as a fresh variable and will play an important role in the proof of cut-free completeness below.

Lemma 14 (Fresh constants)**.** *Let c be a constant and y a variable which does not appear in* Γ, Δ, A. *Assume furthermore that c does not appear in* Γ, Δ.

[7] The rules (μ) and ($\tilde{\mu}$) are not introduction rules, because they do not introduce a formula constructor.

- $\Gamma \vdash A(c) \mid \Delta$ *implies* $\Gamma \vdash A(y) \mid \Delta$.
- $\Gamma \mid A(c) \vdash \Delta$ *implies* $\Gamma \mid A(y) \vdash \Delta$.

Moreover, no further cuts in the derivations on the right-hand side are necessary.

Proof. It follows directly from the lemma just before. □

The fact that Lemma 12 ∼ Lemma 14 need not introduce any new cuts in the derivations on the right-hand side of the implication will be important for the proof of cut-free completeness.

We now show the soundness of $\mathrm{LK}_{\mu\tilde{\mu}}$ with respect to the Kripke semantics. First we need some preparations.

Let $(K, \leq, D, \Vdash_{\mathfrak{s}}, \Vdash_{\perp})$ be a Kripke model. *Associations* are functions from a finite set of free variables to $\bigcup_{w \in K} D(w)$. The letters $\rho, \eta, ...$ vary over associations. Given an association ρ and a free variable x, ρ^{-x} denotes the function obtained from ρ by deleting x from its domain, i.e., $\mathsf{dom}(\rho^{-x}) = \mathsf{dom}(\rho) \backslash \{x\}$. Let $\rho(x \mapsto d)$ denote the function ρ' such that $\rho'(y) = \rho(y)$ if $y \neq x$ and d otherwise.

Let c_0 be a distinguished constant of the language. Given a formula A, let $A[\rho]$ denote the sentence in the extended language with fresh constants for each element of D obtained from A by replacing each free variable x with $\rho(x)$ if $x \in \mathsf{dom}(\rho)$ and with c_0 otherwise. $\Gamma[\rho]$ is the context obtained from Γ by replacing each $A \in \Gamma$ with $A[\rho]$.

We write $w :\Vdash \Gamma$ when w forces all sentences from Γ and $w : \Delta \Vdash$ when w refutes all sentences from Δ.

The intuitive meaning of the following theorem is that if every formula in the assumption is forced, then not all formulae in the conclusion can be refuted.

Theorem 15 (Soundness)**.** *Let A be a formula and Γ, Δ contexts of formulae. In any classical Kripke model $(K, \leq, D, \Vdash_{\mathfrak{s}}, \Vdash_{\perp})$ the following holds: Let $w \in K$ and ρ be an associations with the values from $D(w)$.*

- *If $\Gamma \vdash \Delta$, $w :\Vdash \Gamma[\rho]$ and $w : \Delta[\rho] \Vdash$, then $w :\Vdash_{\perp}$.*
- *If $\Gamma \vdash A|\Delta$, $w :\Vdash \Gamma[\rho]$ and $w : \Delta[\rho] \Vdash$, then $w :\Vdash A[\rho]$.*
- *If $\Gamma|A \vdash \Delta$, $w :\Vdash \Gamma[\rho]$ and $w : \Delta[\rho] \Vdash$, then $w : A[\rho] \Vdash$.*

Proof. One proves easily the three statements simultaneously by induction on the derivations. We demonstrate two non-trivial cases. Suppose $w :\Vdash \Gamma[\rho]$ and $w : \Delta[\rho] \Vdash$.

- Case $(\vee_L)$: Suppose $w' \geq w$ and $w' :\Vdash A[\rho] \vee B[\rho]$. We have to show w' is exploding. But this follows from the fact that $w' : A[\rho] \vee B[\rho] \Vdash_{\mathfrak{s}}$. Note just that $w' : A[\rho] \Vdash$ and $w' : B[\rho] \Vdash$ follow from the I.H. using monotonicity.
- Case $(\exists_L)$: Suppose $w' \geq w$ and $w' :\Vdash (\exists x.A)[\rho]$. We have to show w' is exploding. For this it suffices to show $w' : (\exists x.A(x))[\rho] \Vdash_{\mathfrak{s}}$, i.e., $w'' : A[\rho(x \mapsto d)]) \Vdash$ for all $w'' \geq w'$ and $d \in D(w'')$. Note first that $w'' :\Vdash \Gamma[\rho(x \mapsto d)]$ and $w'' : \Delta[\rho(x \mapsto d)] \Vdash$ by monotonicity because of the freshness of x. By I.H. the claim follows.

□

4. Completeness

As usual when constructively proving completeness of Kripke semantics for a fragment[8] of intuitionistic logic [6, 19, 30], we define a special purpose model, called the *universal model*, built from the deduction system itself. Once we show completeness for this special model, completeness for any model follows (Corollary 19).

Definition 16. *The* Universal classical Kripke model $\mathcal{U}$ *is obtained by setting:*

- *K to the set of pairs* (Γ, Δ) *of contexts of* $LK_{\mu\tilde{\mu}}$*;*
- $(\Gamma, \Delta) \leq (\Gamma', \Delta')$ *iff both* $\Gamma \subseteq \Gamma'$ *and* $\Delta \subseteq \Delta'$*;*
- $(\Gamma, \Delta) : X \Vdash_{\bar{s}}$ *iff the sequent* $\Gamma | X \vdash \Delta$ *is provable without a cut in* $LK_{\mu\tilde{\mu}}$*;*
- $(\Gamma, \Delta) : \Vdash_{\perp}$ *iff the sequent* $\Gamma \vdash \Delta$ *is provable without a cut in* $LK_{\mu\tilde{\mu}}$*;*
- *for any w,* $D(w)$ *is the set of closed terms of* $LK_{\mu\tilde{\mu}}$*.*

Note that the domain function D is a constant function, while in the abstract definition of model we allow for non-constant domain functions because that allows building more counter-models in applications.

Monotonicity of strong refutation on atoms follows from Lemma 12.

Theorem 17 (Cut-Free Completeness for $\mathcal{U}$)**.** *For any sentence A and contexts of sentences* Γ *and* Δ*, the following hold in* $\mathcal{U}$*:*

$$(\Gamma, \Delta) : \Vdash A \quad \Longrightarrow \quad \Gamma \vdash A | \Delta \tag{1}$$

$$(\Gamma, \Delta) : A \Vdash \quad \Longrightarrow \quad \Gamma | A \vdash \Delta \tag{2}$$

Moreover, the derivations on the right-hand side of (1) *and* (2) *are cut-free.*

Proof. We proceed by simultaneously proving the two statements by induction on the complexity of A. When quantifiers are concerned, $A(t)$ has lower complexity than $\exists x.A(x)$ and $\forall x.A(x)$.

The derivation trees in this proof use meta-rules (*) and multi-step derivations ($Contr_L, Contr_L$) in addition to the derivation rules of the calculus from Table 1 in order to make the proofs easier to read.

We also remind the reader that the notion of cut-freeness is the one of $LK_{\mu\tilde{\mu}}$, introduced in the previous section.

Base case for atomic formulae. In the base case we have forcing and refutation on atomic sentences, which by definition reduce to strong refutation on atomic sentences, which by definition reduces just to statements about the deductions in $LK_{\mu\tilde{\mu}}$

(1) Suppose

$$\forall (\Gamma', \Delta') \geq (\Gamma, \Delta), \{\Gamma' | X \vdash \Delta' \Longrightarrow \Gamma' \vdash \Delta'\} \tag{*}$$

where the RHS is cut-free. Then the following holds for $\Gamma' = \Gamma$ and $\Delta' = X, \Delta$:

[8] As previously remarked, there is no constructive proof for full intuitionistic predicate logic.

$$\cfrac{\cfrac{\overline{\Gamma | X \vdash X, \Delta}\ (\mathrm{Ax}_L)}{\Gamma \vdash X, \Delta}\ (*)}{\Gamma \vdash X | \Delta}\ (\mu)$$

(2) Suppose $(\Gamma, \Delta) : X \Vdash$, i.e.,

$$\forall (\Gamma', \Delta') \geq (\Gamma, \Delta), \{(\Gamma', \Delta') :\Vdash X \implies \Gamma' \vdash \Delta'\} \qquad (*)$$

We use $(*)$ to prove $\Gamma, X \vdash \Delta$ without introducing a cut from which the claim follows by the $(\tilde{\mu})$-rule. For this, we need to show $((\Gamma, X), \Delta) :\Vdash X$. Assume $(\Gamma'', \Delta'') \geq ((\Gamma, X), \Delta)$ such that there is a cut-free proof for $\Gamma'' \mid X \vdash \Delta''$. Then by $(Contr_L)$, $\Gamma'' \vdash \Delta''$, that is, (Γ'', Δ'') is exploding.

Base cases for $\top$ *and* $\bot$. Obvious.

Induction case for implication.

(1) Suppose $(\Gamma, \Delta) :\Vdash A_1 \to A_2$, i.e.,

$$\forall (\Gamma', \Delta') \geq (\Gamma, \Delta), \{(\Gamma', \Delta') : A_1 \to A_2 \Vdash_{\bar{s}} \implies \Gamma' \vdash \Delta'\} \qquad (*)$$

We use $(*)$ to prove $\Gamma, A_1 \vdash A_2, \Delta$ without introducing a cut from which the claim follows by the (μ) and $(\to_R)$ rules. We need to show $((\Gamma, A_1), (A_2, \Delta)) : A_1 \to A_2 \Vdash_{\bar{s}}$, i.e. $((\Gamma, A_1), (A_2, \Delta)) :\Vdash A_1$ and $((\Gamma, A_1), (A_2, \Delta)) : A_2 \Vdash$. We show the first one. The second case is similar.

Assume $(\Gamma', \Delta') \geq ((\Gamma, A_1), (\Delta, A_2))$ such that $(\Gamma', \Delta') : A_1 \Vdash_{\bar{s}}$. Using the induction hypothesis we get the following cut-free proof:

$$\frac{\Gamma' \mid A_1 \vdash \Delta'}{\Gamma' \vdash \Delta'}\ (Contr_L)$$

That is, (Γ', Δ') is exploding.

(2) Suppose $(\Gamma, \Delta) : A_1 \to A_2 \Vdash$, i.e.,

$$\forall (\Gamma', \Delta') \geq (\Gamma, \Delta), \{(\Gamma', \Delta') :\Vdash A_1 \to A_2 \implies \Gamma' \vdash \Delta'\} \qquad (*)$$

We use $(*)$ to prove $\Gamma, A_1 \to A_2 \vdash \Delta$ without introducing a cut from which the claim follows by the $(\tilde{\mu})$-rule. We need to show $((\Gamma, A_1 \to A_2), \Delta) :\Vdash A_1 \to A_2$. Assume $(\Gamma'', \Delta'') \geq ((\Gamma, A_1 \to A_2), \Delta)$ such that $(\Gamma'', \Delta'') \Vdash A_1$ and $(\Gamma'', \Delta'') : A_2 \Vdash$. Then, using the induction hypotheses we have the following cut-free proof:

$$\cfrac{\cfrac{\Gamma'' \vdash A_1 \mid \Delta'' \qquad \Gamma'' \mid A_2 \vdash \Delta''}{\Gamma'' \mid A_1 \to A_2 \vdash \Delta''}\ (\to_L)}{\Gamma'' \vdash \Delta''}\ (Contr_L)$$

That is, (Γ'', Δ'') is exploding.

Induction case for $\vee$.

(1) Suppose $(\Gamma,\Delta) :\Vdash A_1 \vee A_2$, i.e.,

$$\forall(\Gamma',\Delta') \geq (\Gamma,\Delta), \{(\Gamma',\Delta') : A_1 \vee A_2 \Vdash_{\overline{S}} \implies (\Gamma',\Delta') \Vdash_{\perp}\} \qquad (*)$$

First we use $(*)$ to show $\Gamma \vdash A_1, A_2, A_1 \vee A_2, \Delta$ without introducing a cut. For this we set $\Gamma' = \Gamma$ and $\Delta' = A_1, A_2, A_1 \vee A_2, \Delta$, that is, we need to show $(\Gamma',\Delta') : A_i \Vdash$ for $i = 1, 2$. Assume $(\Gamma'',\Delta'') \geq (\Gamma',\Delta')$ such that $(\Gamma'',\Delta'') :\Vdash A_i$, then by induction hypotheses $\Gamma'' \vdash A_i \mid \Delta''$. Therefore, by $(Contr_R)$, (Γ'',Δ'') is exploding.
Now we can prove the claim.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\Gamma \vdash A_2, A_1, A_1 \vee A_2, \Delta}{\Gamma \vdash A_2|A_1, A_1 \vee A_2, \Delta}\,(\mu)}{\Gamma \vdash A_1 \vee A_2|A_1, A_1 \vee A_2, \Delta}\,(\vee_L^2)}{\Gamma \vdash A_1, A_1 \vee A_2, \Delta}\,(Contr_R)}{\Gamma \vdash A_1|A_1 \vee A_2, \Delta}\,(\mu)}{\Gamma \vdash A_1 \vee A_2|A_1 \vee A_2, \Delta}\,(\vee_L^1)}{\Gamma \vdash A_1 \vee A_2, \Delta}\,(Contr_R)}{\Gamma \vdash A_1 \vee A_2|\Delta}\,(\mu)$$

(2) The claim follows directly from the $(\vee_L)$-rule and the induction hypothesis because $(\Gamma,\Delta) : A_1 \vee A_2 \Vdash$ implies both $(\Gamma,\Delta) : A_1 \Vdash$ and $(\Gamma,\Delta) : A_2 \Vdash$ by Lemma 7, which does not need to introduce new cuts.

Induction case for $\wedge$.

(1) The claim follows directly from the $(\wedge_R)$-rule and the induction hypotheses because $(\Gamma,\Delta) :\Vdash A_1 \wedge A_2$ implies both $(\Gamma,\Delta) :\Vdash A_1$ and $(\Gamma,\Delta) :\Vdash A_2$, by Lemma 5, which does not need to intruduce new cuts.

(2) Suppose $(\Gamma,\Delta) : A_1 \wedge A_2 \Vdash$, i.e.,

$$\forall(\Gamma',\Delta') \geq (\Gamma,\Delta), \{(\Gamma',\Delta') :\Vdash A_1 \wedge A_2 \implies (\Gamma',\Delta') \Vdash_{\perp}\} \qquad (*)$$

We use $(*)$ to show $\Gamma, A_1 \wedge A_2 \vdash \Delta$ without introducing a cut from which the claim follows by the $(\tilde{\mu})$-rule. By Lemma 5, we need to show $((\Gamma, A_1 \wedge A_2), \Delta) :\Vdash A_i$ for $i = 1, 2$. Assume $(\Gamma'',\Delta'') \geq ((\Gamma, A_1 \wedge A_2), \Delta)$ such that $(\Gamma'',\Delta'') : A_i \Vdash_{\overline{S}}$. Using induction hypotheses we get the following cut-free proof:

$$\cfrac{\cfrac{\Gamma'' \mid A_i \vdash \Delta''}{\Gamma'' \mid A_1 \wedge A_2 \vdash \Delta''}\,(\wedge_L^i)}{\Gamma'' \vdash \Delta''}\,(Contr_L)$$

Therefore, (Γ'',Δ'') is exploding.

Induction case for $\forall$.

(1) Assume $(\Gamma, \Delta) :\Vdash \forall x.A(x)$. Then, by Lemma 5, $(\Gamma, \Delta) :\Vdash A(t)$ for all closed terms. In particular, we have $(\Gamma, \Delta) :\Vdash A(c)$ for some fresh constant c which does not occur in Γ, Δ, A. Using the induction hypothesis we get a cut-free proof of $\Gamma \vdash A(c) \mid \Delta$. By Lemma 14, this implies a cut-free proof of $\Gamma \vdash A(x) \mid \Delta$ for any fresh variable x, so the claim follows.

(2) Suppose $(\Gamma, \Delta) : \forall x.A(x) \Vdash$, i.e.,

$$\forall (\Gamma', \Delta') \geq (\Gamma, \Delta), \{(\Gamma', \Delta') :\Vdash \forall x.A(x) \implies (\Gamma', \Delta') \Vdash_{\perp}\} \qquad (*)$$

We use $(*)$ to show $\Gamma, \forall x.A(x) \vdash \Delta$ without introducing a cut from which the claim follows by the $(\tilde{\mu})$-rule, that is, we need to show $((\Gamma, \forall x.A(x)), \Delta) :\Vdash A(t)$ for any closed term t. Assume $(\Gamma'', \Delta'') \geq ((\Gamma, \forall x.A(x)), \Delta)$ such that $(\Gamma'', \Delta'') : A(t) \Vdash_{\bar{s}}$. Using the induction hypothesis we get the following cut-free proof:

$$\dfrac{\dfrac{\Gamma'' \mid A(t) \vdash \Delta''}{\Gamma'' \mid \forall x.A(x) \vdash \Delta''}\,(\forall_L)}{\Gamma'' \vdash \Delta''}\,(Contr_L)$$

Therefore, (Γ'', Δ'') is exploding.

Induction case for $\exists$.

(1) Suppose $(\Gamma, \Delta) :\Vdash \exists x.A(x)$, i.e.,

$$\forall (\Gamma', \Delta') \geq (\Gamma, \Delta), \{(\Gamma', \Delta') : \exists x.A(x) \Vdash_{\bar{s}} \implies (\Gamma', \Delta') \Vdash_{\perp}\} \qquad (*)$$

We use $(*)$ to show $\Gamma \vdash \exists x.A(x), \Delta$ without introducing a cut from which the claim follows using the (μ)-rule. We need to show $(\Gamma, (\Delta, \exists x.A(x))) : A(t) \Vdash$ for any closed term t.
Assume $(\Gamma'', \Delta'') \geq (\Gamma, (\Delta, \exists x.A(x)))$ such that $(\Gamma'', \Delta'') :\Vdash A(t)$. Using the induction hypothesis we get the following cut-free proof:

$$\dfrac{\dfrac{\Gamma'' \vdash A(t) \mid \Delta''}{\Gamma'' \vdash \exists x.A(x) \mid \Delta''}\,(\exists_R)}{\Gamma'' \vdash \Delta''}\,(Contr_R)$$

Therefore, (Γ'', Δ'') is exploding.

(2) Assume $(\Gamma, \Delta) : \exists x.A(x) \Vdash$, then $(\Gamma, \Delta) : \exists x.A(x) \Vdash_{\bar{s}}$ by Lemma 7. That is, $(\Gamma, \Delta) : A(t) \Vdash$ for all closed terms. In particular, we have $(\Gamma, \Delta) : A(c) \Vdash$ for some fresh constant c which does not occur in Γ, Δ, A. Using induction hypotheses we have a cut-free proof of $\Gamma \mid A(c) \vdash \Delta$. By Lemma 14, this implies a cut-free proof of $\Gamma \mid A(x) \vdash \Delta$ for any fresh variable, so the claim follows.

□

Corollary 18. *For any sentence A and contexts of sentences Γ, Δ, the following hold in $\mathcal{U}$:*

1. *If $A \in \Gamma$ then $(\Gamma, \Delta) :\Vdash A$.*

2. *If* $B \in \Delta$ *then* $(\Gamma, \Delta) : B \Vdash$.

Proof. 1. Assume $A \in \Gamma$, $(\Gamma', \Delta') \geq (\Gamma, \Delta)$ and $(\Gamma', \Delta') : A \Vdash_{\mathfrak{I}}$. Then by Theorem 17, $\Gamma' \mid A \vdash \Delta'$, so we obtain a cut-free proof for $\Gamma' \vdash \Delta'$ using $(Contr_L)$. That is, (Γ', Δ') is exploding.

2. Assume $B \in \Delta$, $(\Gamma', \Delta') \geq (\Gamma, \Delta)$ and $(\Gamma', \Delta') :\Vdash B$. Then by Theorem 17, $\Gamma' \vdash B \mid \Delta'$, so we obtain a cut-free proof for $\Gamma' \vdash \Delta'$ using $(Contr_R)$. That is, (Γ', Δ') is exploding.

□

Corollary 19 (Completeness of Classical Logic)**.** *If in every Kripke model, at every possible world, the sentence A is forced whenever all the sentences of* Γ *are forced and all the sentences of* Δ *are refuted, then there exists a cut-free derivation in* $\mathrm{LK}_{\mu\tilde{\mu}}$ *of the sequent* $\Gamma \vdash A|\Delta$.

Proof. If the hypothesis holds for any Kripke model, so does it hold for $\mathcal{U}$. Theorem 17 and Corollary 18 lead to the claim, since $(\Gamma, \Delta) :\Vdash \Gamma$ and $(\Gamma, \Delta) : \Delta \Vdash$. □

Remark 20. *The following are false, even if reasoning classically.*

- $w :\Vdash A \vee B$ *implies* $w :\Vdash A$ *or* $w :\Vdash B$.
- $w :\Vdash \exists x.A(x)$ *implies* $w :\Vdash A(d)$ *for some* $d \in D(w)$.

Because of the completeness of classical logic with respect to the universal model, the claims correspond to Disjunction property (DP) and Explicit definability property (ED), respectively, which are in general not true in classical logic.

A constructive cut-free completeness theorem can also be used for proof normalisation.

Corollary 21 (Semantic Cut-Elimination)**.** *For all contexts* Γ, Δ *of sentences, if there is a derivation of* $\Gamma \vdash \Delta$*, then there is a cut-free derivation of* $\Gamma \vdash \Delta$.

Proof. From the hypothesis $\Gamma \vdash \Delta$, the soundness theorem applied to $\mathcal{U}$ gives us that there is indeed a cut-free derivation for $\Gamma \vdash \Delta$ because the world (Γ, Δ) forces all formulae of Γ and refutes all formulae of Δ as shown in Corollary 18. □

5. Discussion, Related and Future Work

5.1. Normalisation by Evaluation

The last corollary is at the origin of our work, where we wanted to do a normalisation-by-evaluation (NBE) proof for computational classical logic. The general idea of the NBE method is to use an "evaluation" (soundness) function from the object-language to a constructive meta-language and then use a "reification" (completeness) function from the meta-language back to the object-language. The interpretation of the object-language inside the meta-language, that goes via evaluation/soundness, is usually done using some form of Kripke models.

So far, NBE has been used to show normalisation of various intuitionistic proof systems [5, 11, 2, 1, 28, 30] as well as purely computational calculi [12]. One advantage of taking this approach to that of studying a reduction relation for a proof calculus for classical logic, explicitly as a rewrite system, is that one circumvents both difficulties of rewrite systems and validating equalities arising from η-conversion. For more details on these difficulties the reader is referred to [33], for classical proof systems, and [13] for intuitionistic proof systems. Another advantage is that these kinds of proofs manipulate finite structures only and avoid working with saturated models as, for example, in [31].

Note also that, although as output from the NBE algorithm we get a β-reduced η-long normal form, we proved a weak NBE result, as we did not prove that the output can be obtained from the input by a number of rewrite steps, as it is done in [6].

5.2. Dual Notion of Model

Thanks to the symmetry of the $\mathrm{LK}_{\mu\tilde{\mu}}$ rules for left-distinguished and right-distinguished formulae, it is possible to define a dual notion of model in which:

- "strong *forcing*" is taken as primitive and "refutation" and non-strong "forcing" are defined from it by orthogonality like in Definition 2,
- for the universal model, strong forcing is defined as cut-free provability of *right*-distinguished formulae (instead of left-distinguished ones for strong refutation),

and prove, completely analogously to the proofs presented in this paper, that we have the same soundness and completeness theorems holding.

The reader interested in the computational behaviour of the completeness theorem, should look at its partial Coq formalisation[20]. From that work it follows that the NBE theorem computes the normal forms of proofs in call-by-name discipline. We mention this work because we would like to conjecture that the presented classical Kripke model always gives rise to call-by-name behaviour for proof normalisation, while the dual notion gives rise to call-by-value behaviour. As one of the referees remarked, there is a variety of different strategies for doing proof normalisation, of which call-by-name and call-by-value are the simplest ones to describe, but also the most standard ones. For a general study of cut-elimination strategies that are more complex than call-by-name and call-by-value, the reader is referred to [10].

5.3. Using Intuitionistic Kripke Models on Doubly-Negated Formulae

Although one can define a double-negation interpretation A^* of formulae and use intuitionistic Kripke models and an intuitionistic completeness theorem to obtain a normalisation result, one would have to pass through the chain of inferences

$$\vdash_c A \Longrightarrow\ \vdash_i A^* \Longrightarrow\ \Vdash_i A^* \Longrightarrow\ \vdash_i^{nf} A^* \Longrightarrow\ \vdash_c^{nf} A$$

where "i" stands for "intuitionistic", "c" for "classical" and "nf" for "in normal form", in which how to do the last inference is not obvious. We consider that to be a *detour* since we can prove, simply, the chain of inferences

$$\vdash_c A \Longrightarrow\ \Vdash_c A \Longrightarrow\ \vdash_c^{nf} A$$

The interest in having a direct-style semantics for classical logic is the same as the interest in having a proof calculus for classical logic instead of restricting oneself to an intuitionistic calculus and working with doubly-negated formulae; or, in the theory of programming languages, to having a separate constant *call-cc* instead of writing all programs in continuation-passing style.

Avigad shows in [3] how classical cut-elimination is a special case of intuitionistic one, work which resembles the first chain of inferences of this subsection. However, his work is specialised to "negative" formulae, that is, it is not clear how to extend it to formulae that use $\vee$ and $\exists$.

Finally, we remark that an interpretation through intuitionistic Kripke models and a double-negation interpretation would have to be done in Kripke models with exploding nodes, because of the meta-mathematical results from [21, 26, 27].

5.4. Boolean vs. Kripke Semantics for Classical Logic

We compare Boolean and Kripke semantics in a constructive setting, based on our own observations (which we hope to submit for publication soon) and based on a strand of works in mathematical logic from the 1960s.

Computational Behaviour. The only known constructive completeness proof of classical logic with respect to Boolean models is the one of Krivine[24], who used a double-negation interpretation to translate Gödel's original proof. Krivine's proof was later reworked by Berardi and Valentini [4] to show that its main ingredient is a constructive version of the ultra-filter theorem for countable Boolean algebras. This theorem, however, crucially relies on an enumeration of the members of the algebra (the formulae).

In the work we mentioned as yet to be put into words, a formalisation in constructive type theory of the proof of Berardi and Valentini, we saw that, as a consequence of relying on the linear order, the reduction relation for proof-terms corresponding to implicative formulae is not β-reduction, but an ad hoc reduction relation which depends on the particular way one defines the linear order (enumeration of formulae). As a consequence, there is no clear notion of normal form suggested by the ad hoc reduction relation. The cut-free completeness theorem given in this paper, however, gives rise to a normalisation algorithm which respects the β-reduction relation of the object-language, when the Kripke models are interpreted in a type theory which is based on β-reduction itself.

Expressiveness. We think of classical Kripke model validity as being more expressive, i.e. containing more information, than Boolean model validity. That is indicated by the presented completeness theorem which is both simpler than (constructive) completeness theorems for Boolean models, and manipulates finite structures directly, instead of relying on structures built up by an infinite saturation process.

Also, only after submitting the first version of the present text, we became aware of the work done in the 1960s on using Kripke models to do model theory of classical logic [14]. Although conducted in a *classical* meta-language, the work indicates that it is possible to use Kripke models to express elegantly some cumbersome constructions of model theory, like set theoretic forcing [9, 14]. Indeed, the connection between the two had been spotted already by Kripke [23] and hence the term "forcing" appeared in Kripke semantics. We hope that looking at those kind of constructions inside Kripke models, but this time inside a *constructive* meta-language, might be an interesting venue to finding out the constructive content of techniques of classical model theory.

In this respect, our work can also be seen as a contribution to the field of constructive model theory of classical logic.

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