Some estimates due to Richard Brent applied to the "high jumpers" problem.

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Abstract

We apply the prime k-tuples conjecture and related estimates to the "high-jumpers" problem on gaps between successive primes.

1 Motivation

The following problem was proposed at the 1993 Western Number Theory conference:

93:03 (John Conway & Andrew Odlyzko) Call d a **high-jumper** if d occurs most frequently as the difference of consective primes $\leq x$ for some x (there may be several high-jumpers for a given x).

Example: $x = 11$: 2, 3, 5, 7, 11 gives 1, 2, 2, 4, so $C(11) = \{2\}$.

Conjecture: the only high-jumpers are 4 and the prime factorials 2, 6, 30, 210, 2310, $\cdots$. Can one prove that high-jumpers tend to infinity? That any prime p divides all high-jumpers for $x \geq x_0(p)$?

Remarks: A paragraph of UPINT2 reads as follows:

Victor Meally used the phrase **prime deserts**. He notes that below 373 the commonest gap is 2; below 467 there are 24 gaps of each of 2, 4 and 6; below 563 the commonest gap is 6, as it is between 10^{14} and $10^{14} + 10^8$ and probably also from 2 to 10^{14}. He asks: when does 30 take over as the commonest gap?

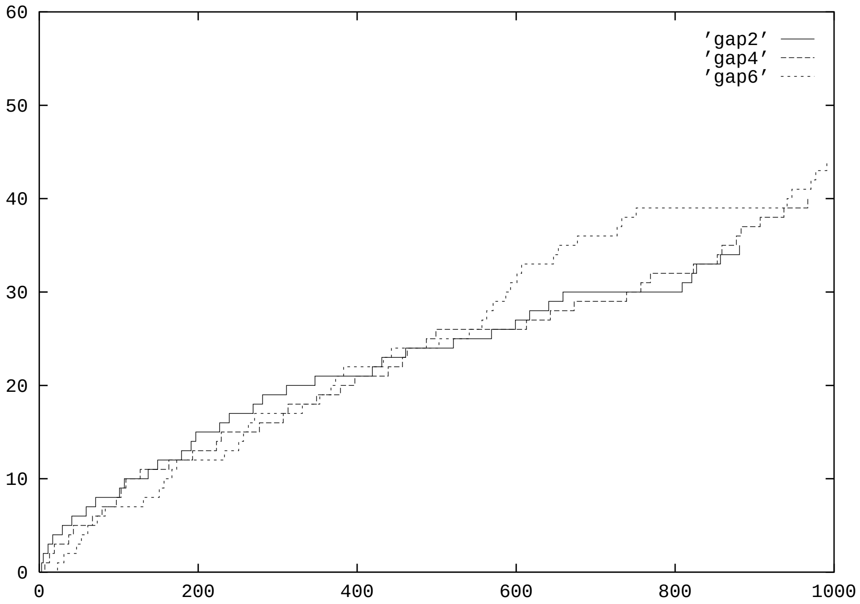
Dan Goldston provides the references:

Harry Nelson, Problem 654, *J. Recreational Math.*, **10**(1977-78) 212.

P. Erdös & E. G. Straus, Remarks on the difference between consecutive primes, *Elem. Math.*, **35**(1980) 115-118.

Erdös & Straus assumed a version of the Hardy-Littlewood prime k-tuple conjecture to show that high-jumpers go to infinity. The proposers can almost certainly show that the conjecture is true if they assume a uniform version of the k-tuple conjecture. This has not been written down yet, though.

Figure 1: Counts of the gaps 2, 4 and 6 between successive primes.



2 Background

We follow the notation of [Bre74]. Let $P_N(r)$ denote the number of primes $p \leq N$ such that $p+2r$ is also prime and let $Q_N(r)$ denote the number of primes $p \leq N$ such that the smallest prime greater than p is $p+2r$. For instance, figure 1 shows Q_N for $N \leq 1000$ and $r = 1$, 2 and 3.

In [HL22], G. H. Hardy and J. E. Littlewood conjectured an asymptotic form of $P_N(1)$ which generalises naturally to the following *prime k-tuples conjecture*. Let m_i be distinct positive integers for $i = 1$ to s. Then the number of primes $p \leq N$ for which $p + 2m_i$ is also prime for all $i \in [1..s]$ is:

$$P_N(m_1, \cdots, m_s) \overset{c}{\sim} C(m_1, \cdots, m_s).\mathrm{li}_{s+1}(N) \qquad \text{(H-L)}$$

as $N \to \infty$, where:

$$C(m_1, \cdots, m_s) = 2^s \prod_q \frac{1 - w_m(q)/q}{(1-1/q)^{s+1}} \; ,$$

q runs over odd primes and $w_m(q)$ counts the distinct residues of $0, m_1, \cdots, m_s$ modulo q. The logarithmic integrals $\mathrm{li}_n(x)$ are defined for $n > 1$ by:

$$\mathrm{li}_n(x) \;=\; \int_0^x \frac{dx}{(\ln x)^n}$$

The singularity in the integrand at $x = 1$ is treated by taking the limit of $\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^x$ as $\varepsilon \to 0$ from above.

B. H. Mayoh ([May68]) suggested the use of inclusion-exclusion to write Q_N in terms of several P_N's and thus obtain an asymptotic form for Q_N subject to H-L:

$$Q_N(r) \;=\; \sum_{k=1}^{r} (-1)^{k-1} \sum_{0<m_1<\cdots<m_{k-1}<r} P_N(m_1, \cdots, m_{k-1}, r)$$

Thus:

$$Q_N(r) \;\overset{c}{\sim}\; \sum_{k=1}^{r} A_{r,k} \mathrm{li}_{k+1}(N)$$

as $N \to \infty$, where:

$$A_{r,k} \;=\; (-1)^{k-1} \sum_{0<m_1<\cdots<m_{k-1}<r} C(m_1, \cdots, m_{k-1}, r)$$

Note that $\mathrm{li}_n(x) \;\in\; \Theta(x/(\ln x)^n)$ hence (subject to H-L):

$$Q_N(r) \;=\; A_{r,1}.\mathrm{li}_2(N) \;+\; o(\mathrm{li}_2(N))$$

and that:

$$A_{r,1} \;=\; C(r) \;=\; 2c_2 \prod_{q|r} (q-1)/(q-2)$$

where c_2 is the twin-prime constant (see below). Thus H-L implies that a gap $2r$ which is a prime factorial will eventually become more frequent than all smaller gaps. Furthermore the high-jumpers go to infinity, as found by Erdös and Straus.

3 Contribution

In [Bre74] Richard Brent compared $Q_N(r)$ with its asymptotic form given above, for various r. He observed a good agreement and found that the lower order terms improve the agreement, at least in the range he examined i.e., $N \leq 10^9$.

We estimate $Q_N(r)$ using the asymptotic form. We first compute the Hardy-Littlewood constants:

$$c_k = \prod_{q>k} \frac{1-k/q}{(1-1/q)^k}$$

where q runs over odd primes. John W. Wrench, Jr. showed in [Wre61] that this could be done for c_2 with the prime zeta function $P(s) = \sum_p p^{-s}$ where p runs over all primes. We generalise Wrench's method in the obvious way:

$$\ln c_k = -\sum_{j=2}^{\infty} \frac{k^j - k}{j} \left(P(j) - \sum_{p \leq k} p^{-j} \right)$$

where again p runs over primes. Hans Riesel describes in [Rie94] how to compute the prime zeta function via Möbius inversion as:

$$P(s) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \ln \zeta(ks)$$

Even with these formulae, the Hardy-Littlewood constants are rather difficult to compute. Nevertheless we obtained them to high precision (100 decimals) in a reasonable amount of time (about 2 minutes each on a Sparc v8) with the `pari/gp` calculator of Henri Cohen et al. Table 1 shows the results with 50 decimals.

Next we computed the integers $T_{r,k}$ which relate the Hardy-Littlewood constants to the $A_{r,k}$ as described by Brent.

Finally, we need to calculate the logarithmic integrals. We derived the following generalization of the known series for li_1 by elementary methods (integration of a series term by term and use of a recursion found by partial integration):

$$\mathrm{li}_n(x) \;=\; (\gamma - H_{n-1} + \ln\ln x)/(n-1)! + \sum_{k\geq 0 \wedge k\neq n-1} \frac{(\ln x)^{k-n+1}}{k!(k-n+1)}$$

where γ is Euler's gamma ($\simeq 0.577$) and H_n is the harmonic sum $\sum_{k=1}^{n} 1/k$.

Now we can find for which N the estimate for $Q_N(15)$ passes that for $Q_N(3)$. We find that it occurs at approximately $1.7 * 10^{35}$. Note that this result is at best a "guesstimate" of where the gap 30 passes the gap 6, since error bounds on the estimates are not known.

Table 1: Hardy-Littlewood constants to 50 decimals

c_2	=	0.66016 18158 46869 57392 78121 10014 55577 84326 23360 28473+
c_3	=	0.63516 63546 04271 20720 66965 91272 52241 73420 65687 33237+
c_4	=	0.30749 48787 58327 09312 33544 86071 07685 30221 78519 95066+
c_5	=	0.40987 48850 88236 47447 87812 12337 95527 78963 58013 25495-
c_6	=	0.18661 42973 58358 39665 69248 47944 18833 78400 73944 94559-
c_7	=	0.36943 75103 86498 68932 31907 49876 75077 70553 72913 89303+
c_8	=	0.23241 93345 86716 54620 61302 12670 66423 16017 58032 64641-
c_9	=	0.12017 12067 74741 72146 99894 64638 47738 34654 94082 65088-
c_{10}	=	0.04180 40508 12181 65710 39748 72842 46412 05882 81078 159110-
c_{11}	=	0.09453 08285 13547 15843 37971 55991 07208 84456 47523 053632-
c_{12}	=	0.03539 32598 44463 70442 74555 96801 94105 85290 48910 978298+
c_{13}	=	0.11170 39095 63244 95377 12104 54656 08232 51503 03622 12159-
c_{14}	=	0.06284 46339 43607 42176 34598 50911 39350 53388 62666 290321-
c_{15}	=	0.02924 41621 25091 30867 18534 29424 46031 81891 89081 789260-
c_{16}	=	0.00922 81011 53092 67664 26576 54374 72155 09035 30929 0851726-

References

[Bre74] Richard P. Brent. *The Distribution of Small Gaps Between Successive Primes.* Mathematics of Computation, 28:315–324, 1974.

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