Proof assistants in computer science research

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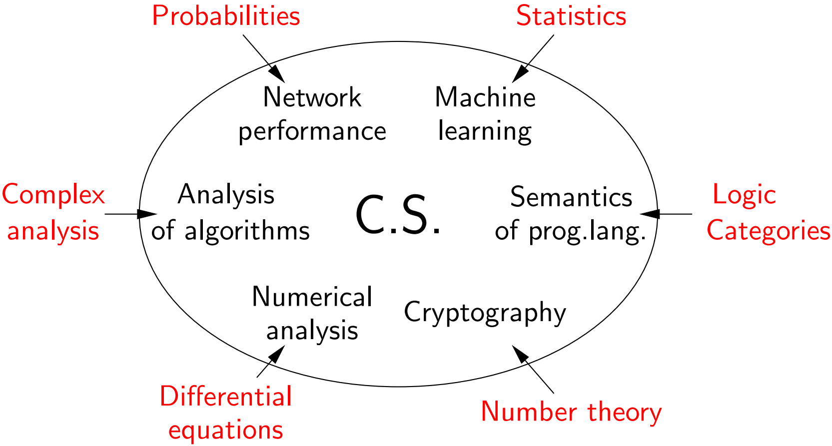
Semantics of proofs and certified mathematics, 2014-04-22

Mathematics in computer science



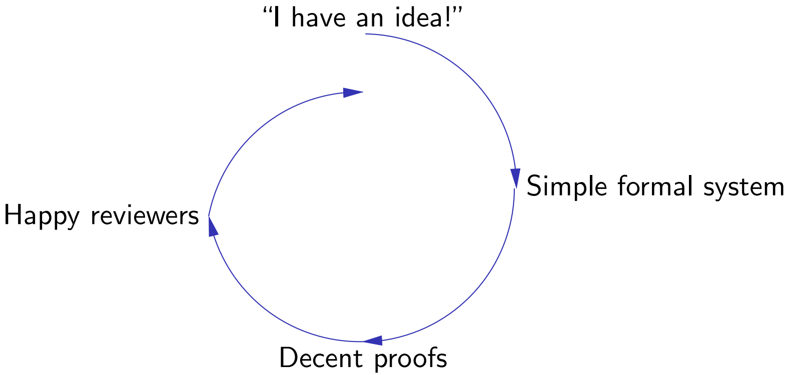
Mathematical models and proofs are essential in many areas of C.S., for validation as well as for discovery.

A tension

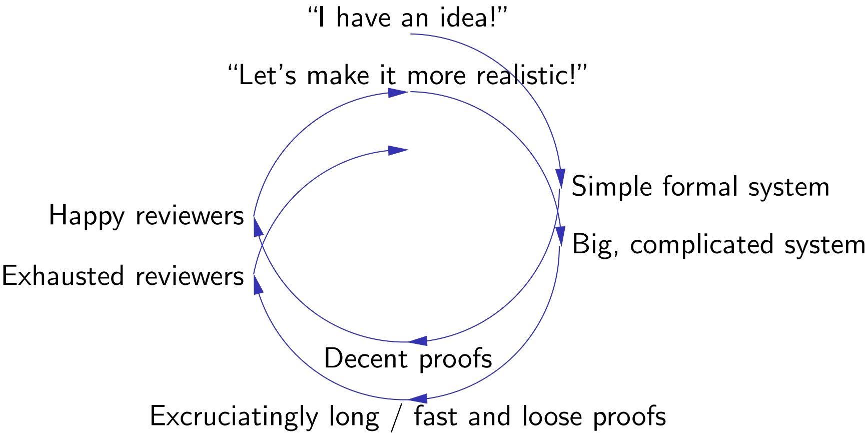
Mathematics ⟷ Engineering

Elegant, uncluttered abstractions ⟷ Practical, overly detailed artifacts

A vicious circle



A vicious circle



Machine assistance to the rescue?

Proofs written by computer scientists are boring:
they read as if the author is programming the reader.

(John C. Mitchell)

Who said that the reader must be human?
Proofs can and should be checked by computers.

(The proof assistant community)

In this talk

Three short stories where the use of proof assistants enables C.S. research to scale properly:

- Programming languages metatheory (POPLmark)
- Deductive verification of critical software (seL4)
- Formally-verified compilation (CompCert)

Part I

The metatheory of programming languages

Formally defining a programming language

Syntax

(what do programs look like?)

Dynamic semantics

(how programs execute? what do they compute?)

Type system / Static semantics

(what are well-formed programs?)

A trivial language: simply-typed λ-calculus

Syntax: $\mathit{Expr} \ni a ::= N \mid x \mid \lambda x.a. \mid a_1 \; a_2$

Dynamic semantics: $(\lambda x.a_1) \; a_2 \to a_1[x \leftarrow a_2]$

Type system:

$$\Gamma \vdash N : \mathtt{nat} \qquad \frac{(x : \tau) \in \Gamma}{\Gamma \vdash x : \Gamma(x)} \qquad \frac{\Gamma,\; x : \tau \vdash a : \tau'}{\Gamma \vdash \lambda x.a : \tau \to \tau'}$$

$$\frac{\Gamma \vdash a_1 : \tau \to \tau' \qquad \Gamma \vdash a_2 : a_2 : \tau}{\Gamma \vdash a_1 \; a_2 : \tau'}$$

The metatheory of a programming language

A pretentious word to refer to important properties that hold for all well-typed programs, e.g.

- Type soundness: execution never crashes on an undefined computation

$$\forall a. \quad a \to \cdots \not\to N \; b$$

- Normalization: execution always terminates

$$\forall a, \exists b, \quad a \to \cdots \to b \not\to$$

Plus: decidability of type-checking; existence of principal types; etc.

Scaling up: System Fsub

Adds support for polymorphism and subtyping as in OO languages.

$$\frac{(x : \tau) \in \Gamma}{\Gamma \vdash x : \Gamma(x)} \qquad \frac{\Gamma,\ x : \tau \vdash a : \tau'}{\Gamma \vdash \lambda x.a : \tau \to \tau'}$$

$$\frac{\Gamma \vdash a_1 : \tau \to \tau' \qquad \Gamma \vdash a_2 : a_2 : \tau}{\Gamma \vdash a_1\ a_2 : \tau'} \qquad \frac{\Gamma \vdash a : \tau \qquad \Gamma \vdash \tau <: \tau'}{\Gamma \vdash a : \tau'}$$

$$\frac{\Gamma,\ X <: \tau \vdash a : \tau'}{\Gamma \vdash \Lambda X <: \tau.a : \forall X <: \tau.\ \tau'}$$

$$\frac{\Gamma \vdash a : \forall X <: \tau_1.\ \tau_2 \qquad \Gamma \vdash \tau <: \tau_1}{\Gamma \vdash a[\tau] : \tau_2[X \leftarrow \tau]}$$

Scaling up: System Fsub

Subtyping rules:

$$\Gamma \vdash \tau <: \top \qquad \Gamma \vdash X <: X \qquad \frac{\Gamma \vdash \tau_1' <: \tau_1 \quad \Gamma \vdash \tau_2 <: \tau_2'}{\Gamma \vdash \tau_1 \to \tau_2 <: \tau_1' \to \tau_2'}$$

$$\frac{\Gamma \vdash \tau_1' <: \tau_1 \quad \Gamma, X <: \tau_1 \vdash \tau_2 <: \tau_2'}{\Gamma \vdash \forall X <: \tau_1.\ \tau_2 <: \forall X <: \tau_1'.\ \tau_2'}$$

$$\frac{(X <: \tau') \in \Gamma \quad \Gamma \vdash \tau' <: \tau}{\Gamma \vdash X <: \tau}$$

Mind the gap!

Kernel Fsub: type-checking is decidable.

$$\frac{\Gamma, X <: \tau \vdash \tau_2 <: \tau_2'}{\Gamma \vdash \forall X <: \tau.\ \tau_2 <: \forall X <: \tau.\ \tau_2'}$$

Full Fsub: type-checking is undecidable.

$$\frac{\Gamma \vdash \tau_1' <: \tau_1 \qquad \Gamma, X <: \tau_1 \vdash \tau_2 <: \tau_2'}{\Gamma \vdash \forall X <: \tau_1.\ \tau_2 <: \forall X <: \tau_1'.\ \tau_2'}$$

Growing pains

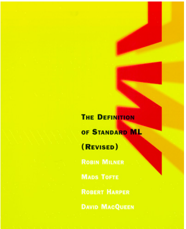
So many type systems to explore!

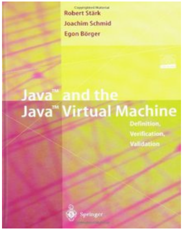
- Type more features: imperative, object-orientation, concurrency, distribution, ...
- Type them more precisely: polymorphism, type abstraction, refinement types, dependent types, ...
- Extend type safety to low-level languages: virtual machines, assembly.

Metatheory proofs become intractable:

- Large case analyses (e.g. 20-page appendix).
- Interesting cases are lost in a sea of routine cases.
- The patience of reviewers is exhausted.

The Grail: formalizing real-world programming languages





Book-sized specifications; hundreds of inference rules.

Very little can be proved about them.

The POPLmark challenge

B.E. Aydemir, B.C. Pierce, S. Weirich, S. Zdancewic, et al, 2005

A vision:

> *How close are we to a world where every paper on programming languages is accompanied by an electronic appendix with machine-checked proofs?*

A challenge: mechanize the metatheory of Fsub in the proof assistant of your choice.

The result: 15 solutions (+/- complete), in 7 proof assistants.

The good news

All solutions handle part 1 of the challenge, which is the most difficult proof of Fsub's metatheory.

Theorem (Transitivity of subtyping)

If $\Gamma \vdash \tau_1 <: \tau_2$ *and* $\Gamma \vdash \tau_2 <: \tau_3$ *then* $\Gamma \vdash \tau_1 <: \tau_3$*.*

Proof: induction on the size of τ_2 and mutual induction with

Theorem (Narrowing)

If $\Gamma, X <: \tau_1, \Delta \vdash \tau <: \tau'$ *and* $\Gamma \vdash \tau_2 <: \tau_3$*, then* $\Gamma, X <: \tau_2, \Delta \vdash \tau <: \tau'$*.*

Moreover, the mechanized proofs are not much bigger than a detailed pencil-and-paper proof.

The bad news

All solutions have a hard time dealing with variable bindings and invariance by renaming of bound variables (α-equivalence).

$$\forall X.\ P(X)$$

$$\sum_{i=0}^{i=n} i^2$$

$$\int_0^\infty f(x)dx$$

$$x \mapsto e^{-x}$$

The bad news

All solutions have a hard time dealing with variable bindings and invariance by renaming of bound variables (α-equivalence).

$$\begin{aligned}
\forall X.\ P(X) &= \forall Y.\ P(Y) \\
\sum_{i=0}^{i=n} i^2 &= \sum_{j=0}^{j=n} j^2 \\
\int_0^\infty f(x)dx &= \int_0^\infty f(t)dt \\
x \mapsto e^{-x} &= z \mapsto e^{-z} \\
&\neq e \mapsto e^{-e} \qquad \text{name capture!}
\end{aligned}$$

Approaches to variable bindings

Either: special support in the logic and proof assistant:

- Nominal logic (Pitts et al);
- Higher-Order Abstract Syntax in Twelf.

Or: special encodings so that α-equivalence is equality:

- de Bruijn indices;
- locally nameless techniques;
- parametric HOAS;
- etc.

No “best” approach yet.

Definitions and statements do not look exactly like on paper.

Mechanizing large programming languages

Several successes for practically-important languages:

- Java and the JVM (Klein, Lochbihler, Nipkow)
- Standard ML (Crary, Harper)
- C (Norrish, Leroy, Krebbers)
- Javascript (Gardner et al)
- x86 machine language (Morrisett et al, Myreen, Benton)

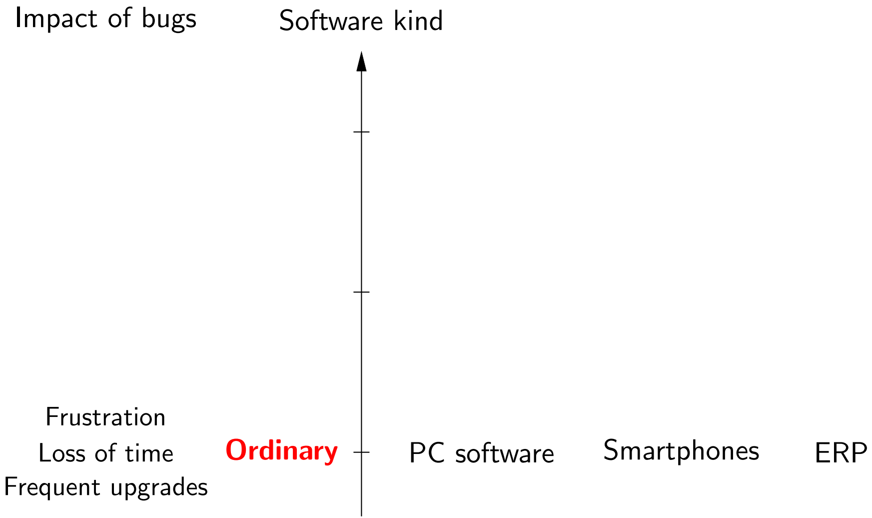
These formalizations are usable (and used) to prove specific programs as well as general metatheoretic results.

15% of POPL submissions now include a machine-checked proof.

Part II

Formal verification of critical software

The software reliability landscape



The software reliability landscape

Impact of bugs	Software kind			
Loss of money Bad PR Getting sued	**Sensitive**	Data security	Network security	Privacy on Internet
Frustration Loss of time Frequent upgrades	**Ordinary**	PC software	Smartphones	ERP

The software reliability landscape

Impact of bugs	Software kind				
Someone dies All of the below	**Critical**	Railways	Medical	Nuclear plants	Airplanes
Loss of money Bad PR Getting sued	**Sensitive**	Data security	Network security	Privacy on Internet	
Frustration Loss of time Frequent upgrades	**Ordinary**	PC software	Smartphones	ERP	

Validating critical software

> *Program testing can be used to show the presence of bugs,*
> *but never to show their absence!*
>
> *(E.W.Dijkstra, 1972)*

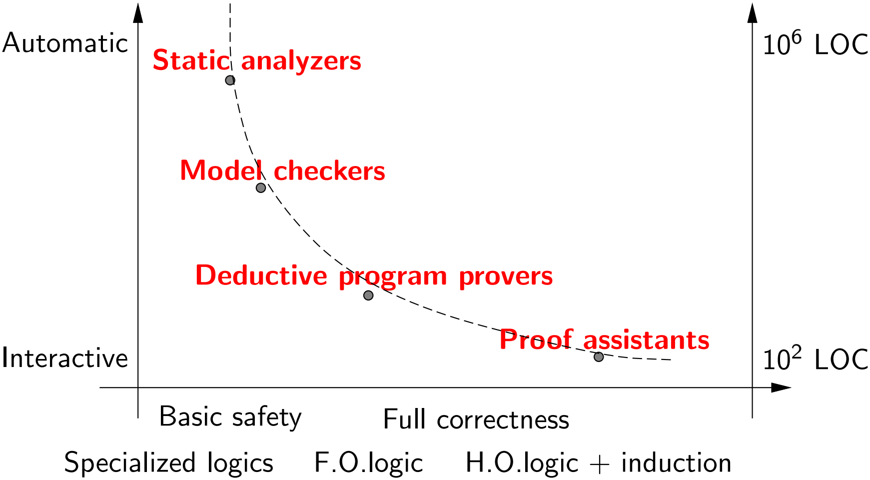
Dominant approach: testing.

Alternative on the rise: tool-assisted formal verification:
verify, possibly infer, properties that hold of all possible executions of a program.

Used in some industrial contexts (airplanes, railways)

- To obtain independent guarantees (besides testing).
- To obtain stronger guarantees (than with testing).
- To save time and money (rigorous testing is expensive).

A panorama of verification tools



Example: computing prime numbers

Knuth, *The Art of Computer Programming*, vol.1

```
    int a[] = new int[n];
    a[0] = 2;
  loop:
    for (int i = 1, m = 3; i < n; m = m + 2) {
        int j = 0;
        while (j < i ∧ a[j] <= √m ) {
            if (a[j] divides m) continue loop;
            j = j + 1;
        }
        a[i] = m; i = i + 1;
    }
```

Goal: compute the first n prime numbers.

Algorithm: try successive odd numbers m, striking out those divisible by primes already found.

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      }
      a[i] = m; i = i + 1;
  }
```

Static analyzer: can infer $1 \leq i < n$ and $0 \leq j < i$ inside the loop, hence array accesses are safe (within bounds).

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```

Automatic program prover: can prove partial correctness if the user provides detailed loop invariants and simple axioms about primality and divisibility. (Termination is harder to prove.)

Example: computing prime numbers

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```
  int a[] = new int[n];
  a[0] = 2;
loop:
  for (int i = 1, m = 3; i < n; m = m + 2) {
    /* invariant:
      ∀k,  0 ≤ k < i  ⇒  isprime(a[k])
      ∀p,  2 ≤ p < m ∧ isprime(p)  ⇒  ∃k,  0 ≤ k < i ∧ a[k] = p
      ∀k, m,  0 ≤ k < j < i  ⇒  a[k] < a[j]
    */
```

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      }
      ...
```

Knuth's cunning optimization: the test `j < i` is redundant and can be omitted. Can you see why?

Example: computing prime numbers

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          j = j + 1;
      }
      ...
```

Knuth's cunning optimization: the test `j < i` is redundant and can be omitted. Can you see why? Because of Bertrand's postulate!

Theorem (Chebychev)

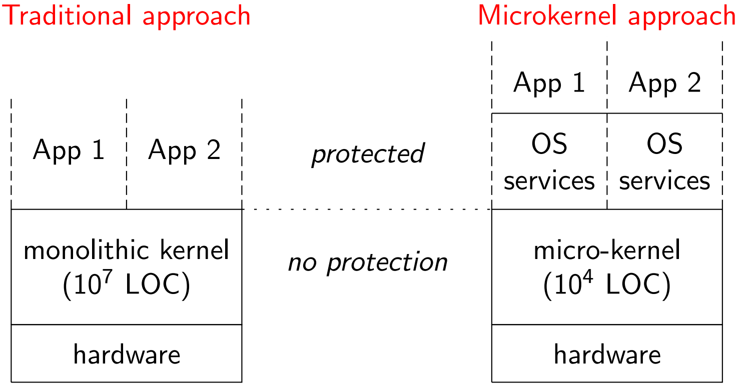
For all $n \geq 1$, there exists a prime p in $[n, 2n]$.

(Coq proof: Laurent Théry, 2002.)

Scaling up: the seL4 verified microkernel

(G. Klein et al, NICTA)

The security core of an operating system.



Verifying seL4 with Isabelle/HOL

(G. Klein et al, NICTA; ACM TOPLAS 32(1), 2014)

Haskell
prototype

Verifying seL4 with Isabelle/HOL

(G. Klein et al, NICTA; ACM TOPLAS 32(1), 2014)

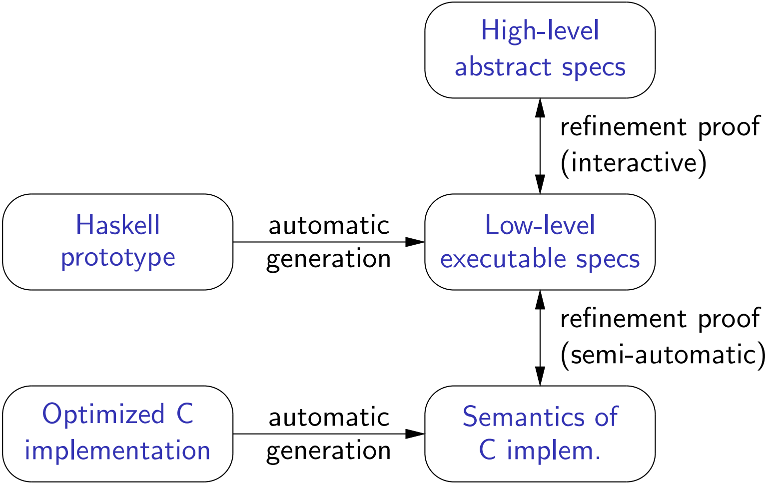
High-level
abstract specs

Haskell
prototype

Optimized C
implementation

Verifying seL4 with Isabelle/HOL

(G. Klein et al, NICTA; ACM TOPLAS 32(1), 2014)



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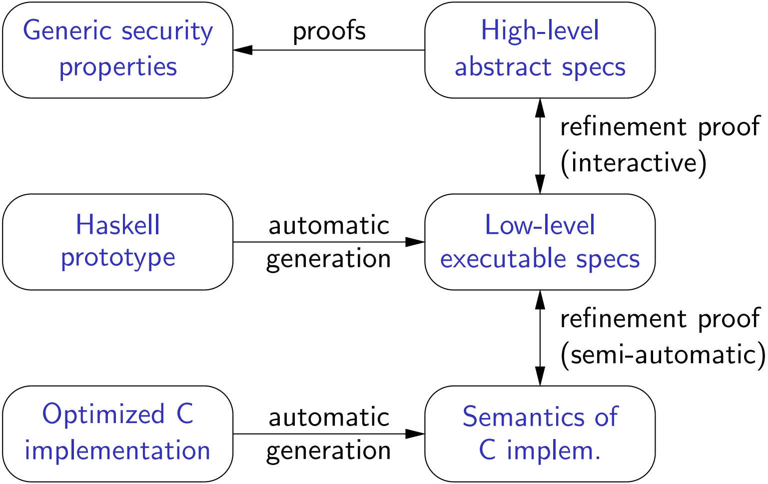


seL4: a milestone

A dream comes true.
($\approx$ 15 unsuccessful OS verification projects since the late 1970's).

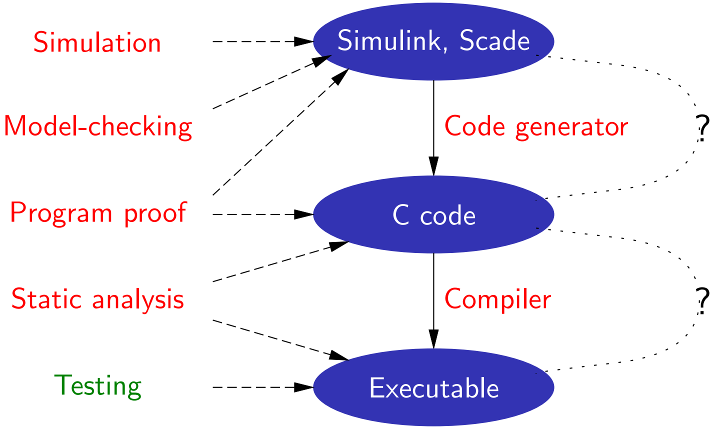
No compromise on the performance of the OS.
($\approx$ 80% of the speed of the fastest unverified L4-style kernel.)

The largest deductive verification of a software system ever:
20 person.year, 200+ KLOC proofs.

Part III

Formally-verified compilation

Trust in software verification



The unsoundness risk: Are verification tools semantically sound?

The miscompilation risk: Are compilers semantics-preserving?

Miscompilation happens

NULLSTONE isolated defects [in integer division] in twelve of twenty commercially available compilers that were evaluated.

`http://www.nullstone.com/htmls/category/divide.htm`

We tested thirteen production-quality C compilers and, for each, found situations in which the compiler generated incorrect code for accessing volatile variables.

E. Eide & J. Regehr, EMSOFT 2008

To improve the quality of C compilers, we created Csmith, a randomized test-case generation tool, and spent three years using it to find compiler bugs. During this period we reported more than 325 previously unknown bugs to compiler developers. Every compiler we tested was found to crash and also to silently generate wrong code when presented with valid input.

X. Yang, Y. Chen, E. Eide & J. Regehr, PLDI 2011

An example of optimizing compilation

```
double dotproduct(int n, double * a, double * b)
{
    double dp = 0.0;
    int i;
    for (i = 0; i < n; i++) dp += a[i] * b[i];
    return dp;
}
```

Compiled with a good compiler, then manually decompiled back to C…

```
double dotproduct(int n, double a[], double b[]) {
     dp = 0.0;
     if (n <= 0) goto L5;
     r2 = n - 3; f1 = 0.0; r1 = 0; f10 = 0.0; f11 = 0.0;
     if (r2 > n || r2 <= 0) goto L19;
     prefetch(a[16]); prefetch(b[16]);
     if (4 >= r2) goto L14;
     prefetch(a[20]); prefetch(b[20]);
     f12 = a[0]; f13 = b[0]; f14 = a[1]; f15 = b[1];
     r1 = 8; if (8 >= r2) goto L16;
L17: f16 = b[2]; f18 = a[2]; f17 = f12 * f13;
     f19 = b[3]; f20 = a[3]; f15 = f14 * f15;
     f12 = a[4]; f16 = f18 * f16;
     f19 = f29 * f19; f13 = b[4]; a += 4; f14 = a[1];
     f11 += f17; r1 += 4; f10 += f15;
     f15 = b[5]; prefetch(a[20]); prefetch(b[24]);
     f1 += f16; dp += f19; b += 4;
     if (r1 < r2) goto L17;
L16: f15 = f14 * f15; f21 = b[2]; f23 = a[2]; f22 = f12 * f13;
     f24 = b[3]; f25 = a[3]; f21 = f23 * f21;
     f12 = a[4]; f13 = b[4]; f24 = f25 * f24; f10 = f10 + f15;
     a += 4; b += 4; f14 = a[8]; f15 = b[8];
     f11 += f22; f1 += f21; dp += f24;
L18: f26 = b[2]; f27 = a[2]; f14 = f14 * f15;
     f28 = b[3]; f29 = a[3]; f12 = f12 * f13; f26 = f27 * f26;
     a += 4; f28 = f29 * f28; b += 4;
     f10 += f14; f11 += f12; f1 += f26;
     dp += f28;  dp += f1; dp += f10; dp += f11;
     if (r1 >= n) goto L5;
L19: f30 = a[0]; f18 = b[0]; r1 += 1; a += 8; f18 = f30 * f18; b += 8;
     dp += f18;
     if (r1 < n) goto L19;
L5:  return dp;
L14: f12 = a[0]; f13 = b[0]; f14 = a[1]; f15 = b[1]; goto L18;
}
```

```
L17: f16 = b[2]; f18 = a[2]; f17 = f12 * f13;
     f19 = b[3]; f20 = a[3]; f15 = f14 * f15;
     f12 = a[4]; f16 = f18 * f16;
     f19 = f29 * f19; f13 = b[4]; a += 4; f14 = a[1];
     f11 += f17; r1 += 4; f10 += f15;
     f15 = b[5]; prefetch(a[20]); prefetch(b[24]);
     f1 += f16; dp += f19; b += 4;
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```

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double dotproduct(int n, double a[], double b[]) {
     dp = 0.0;
     if (n <= 0) goto L5;
     r2 = n - 3; f1 = 0.0; r1 = 0; f10 = 0.0; f11 = 0.0;
     if (r2 > n || r2 <= 0) goto L19;
     prefetch(a[16]); prefetch(b[16]);
     if (4 >= r2) goto L14;
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L16: f15 = f14 * f15; f21 = b[2]; f23 = a[2]; f22 = f12 * f13;
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     f28 = b[3]; f29 = a[3]; f12 = f12 * f13; f26 = f27 * f26;
     a += 4; f28 = f29 * f28; b += 4;
     f10 += f14; f11 += f12; f1 += f26;
     dp += f28;  dp += f1; dp += f10; dp += f11;
     if (r1 >= n) goto L5;
L19: f30 = a[0]; f18 = b[0]; r1 += 1; a += 8; f18 = f30 * f18; b += 8;
     dp += f18;
     if (r1 < n) goto L19;
L5:  return dp;
L14: f12 = a[0]; f13 = b[0]; f14 = a[1]; f15 = b[1]; goto L18;
}
```

Addressing miscompilation

Best industrial practices: more testing; manual reviews of generated assembly code; turn optimizations off; . . .

A more radical solution: why not formally verify the compiler itself?

After all, compilers have simple specifications:

> *If compilation succeeds, the generated code should behave as prescribed by the semantics of the source program.*

As a corollary, we obtain:

> *Any safety property of the observable behavior of the source program carries over to the generated executable code.*

An old idea. . .

John McCarthy
James Painter[1]

CORRECTNESS OF A COMPILER FOR ARITHMETIC EXPRESSIONS[2]

1. **Introduction.** This paper contains a proof of the correctness of a simple compiling algorithm for compiling arithmetic expressions into machine language.

The definition of correctness, the formalism used to express the description of source language, object language and compiler, and the methods of proof are all intended to serve as prototypes for the more complicated task of proving the correctness of usable compilers. The ultimate goal, as outlined in references [1], [2], [3] and [4] is to make it possible to use a computer to check proofs that compilers are correct.

Mathematical Aspects of Computer Science, 1967

An old idea...

3

Proving Compiler Correctness in a Mechanized Logic

R. Milner and R. Weyhrauch

Computer Science Department
Stanford University

Abstract

We discuss the task of machine-checking the proof of a simple compiling algorithm. The proof-checking program is LCF, an implementation of a logic for computable functions due to Dana Scott, in which the abstract syntax and extensional semantics of programming languages can be naturally expressed. The source language in our example is a simple ALGOL-like language with assignments, conditionals, whiles and compound statements. The target language is an assembly language for a machine with a pushdown store. Algebraic methods are used to give structure to the proof, which is presented only in outline. However, we present in full the expression-compiling part of the algorithm. More than half of the complete proof has been machine checked, and we anticipate no difficulty with the remainder. We discuss our experience in conducting the proof, which indicates that a large part of it may be automated to reduce the human contribution.

Machine Intelligence (7), 1972.

The CompCert project

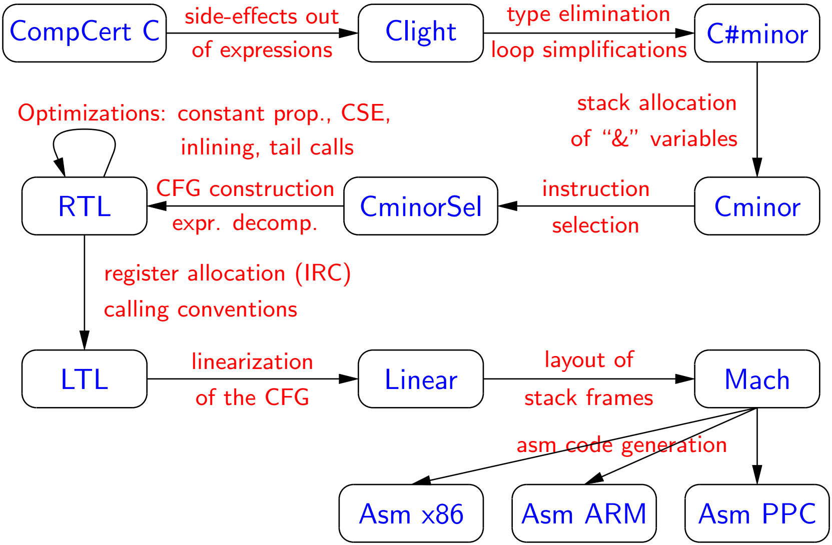
(X.Leroy, S.Blazy, et al)

Develop and prove correct a realistic compiler, usable for critical embedded software.

- Source language: a very large subset of C99.
- Target language: PowerPC/ARM/x86 assembly.
- Generates reasonably compact and fast code
  $\Rightarrow$ careful code generation; some optimizations.

Note: compiler written from scratch, along with its proof; not trying to prove an existing compiler.

The formally verified part of the compiler



Formally verified using Coq

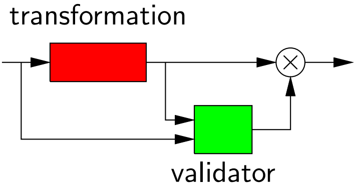
The correctness proof (semantic preservation) for the compiler is entirely machine-checked, using the Coq proof assistant.

```
Theorem transf_c_program_preservation:
  forall p tp beh,
  transf_c_program p = OK tp ->
  program_behaves (Asm.semantics tp) beh ->
  exists beh', program_behaves (Csem.semantics p) beh'
            /\ behavior_improves beh' beh.
```

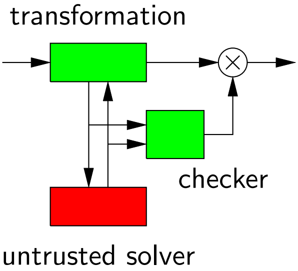
Compiler verification patterns (for each pass)

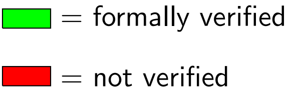
Verified transformation

Verified translation validation

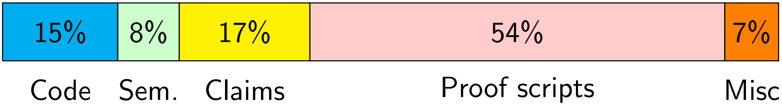


External solver with verified validation





Proof effort



100,000 lines of Coq.

Including 15000 lines of “source code” ($\approx$ 60,000 lines of Java).

6 person.years

Low proof automation (could be improved).

Programmed (mostly) in Coq

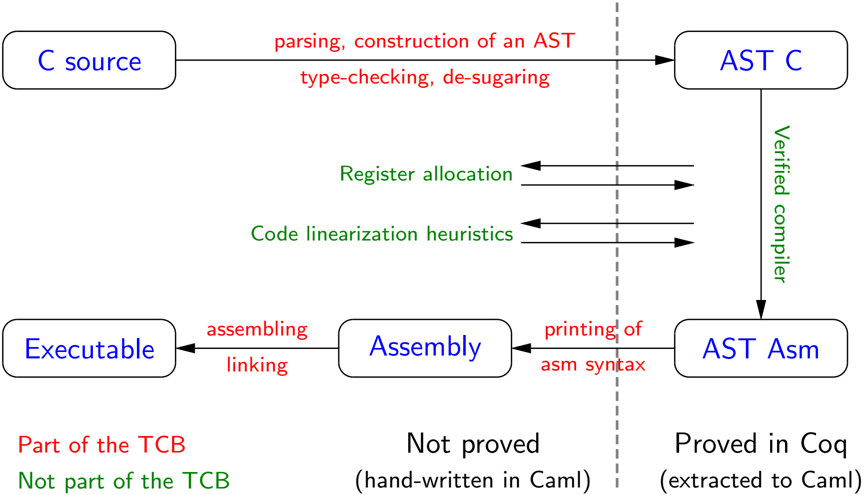
All the verified parts of the compiler are programmed directly in Coq's specification language, using pure functional style.

- Monads to handle errors and mutable state.
- Purely functional data structures.

Coq's extraction mechanism produces executable Caml code from these specifications.

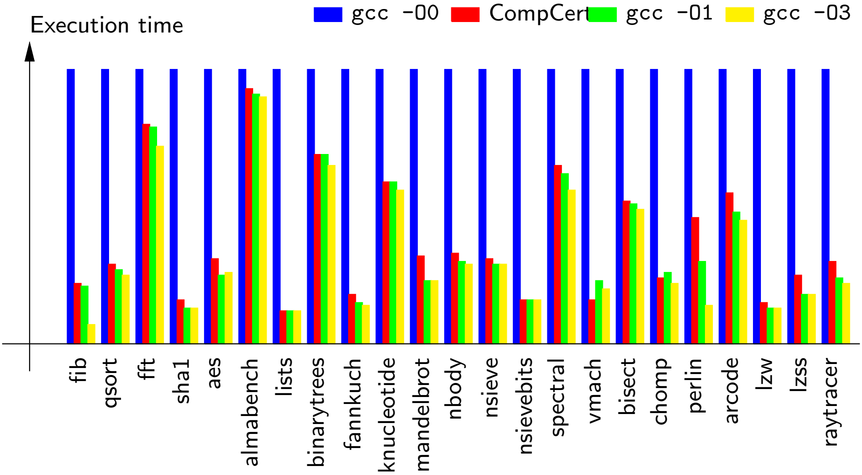
Claim: purely functional programming is the shortest path to writing and proving a program.

The whole Compcert compiler



Performance of generated code

(On a Power 7 processor)



A tangible increase in quality

The striking thing about our CompCert results is that the middleend bugs we found in all other compilers are absent. As of early 2011, the under-development version of CompCert is the only compiler we have tested for which Csmith cannot find wrong-code errors. This is not for lack of trying: we have devoted about six CPU-years to the task. The apparent unbreakability of CompCert supports a strong argument that developing compiler optimizations within a proof framework, where safety checks are explicit and machine-checked, has tangible benefits for compiler users.

X. Yang, Y. Chen, E. Eide, J. Regehr, PLDI 2011

Part IV

Conclusions

In closing. . .

Proof assistants enable unprecedented scaling in many areas of computer science:

- in size and realism of the formal systems considered;
- in mathematical assurance.

Additional benefits:

- Make research papers easier to write and to read.
- Give a second chance to students/engineers/scientists who are insecure in their abilities to do mathematics on paper.
- Facilitate collaborative work of the free software kind.

Some points to keep in mind

Mechanized proofs do not eliminate errors, they reduce the errors to the definitions and statements of theorems.

Proof assistants are addictive and a huge time sink.

Proper engineering of specifications and proofs is crucial.

Fragmentation of the community around multiple theorem provers. (Just like programming languages.)

Mechanized proofs require maintenance and proper archival.

Go forth
and
mechanize!

For more information on the projects presented:
POPLmark: `http://www.seas.upenn.edu/~plclub/poplmark`
seL4: `http://ssrg.nicta.com.au/projects/seL4/`
CompCert: `http://compcert.inria.fr/`