Finite Developments in the λ-calculus

Part 2

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A labeled lambda-calculus (1/3)

- Give names to redexes and to (some) subterms
- make names consistent with permutation equivalence.

$$M, N, \ldots ::= x \mid MN \mid \lambda x.M \mid M^\alpha$$

- Conversion rule is:

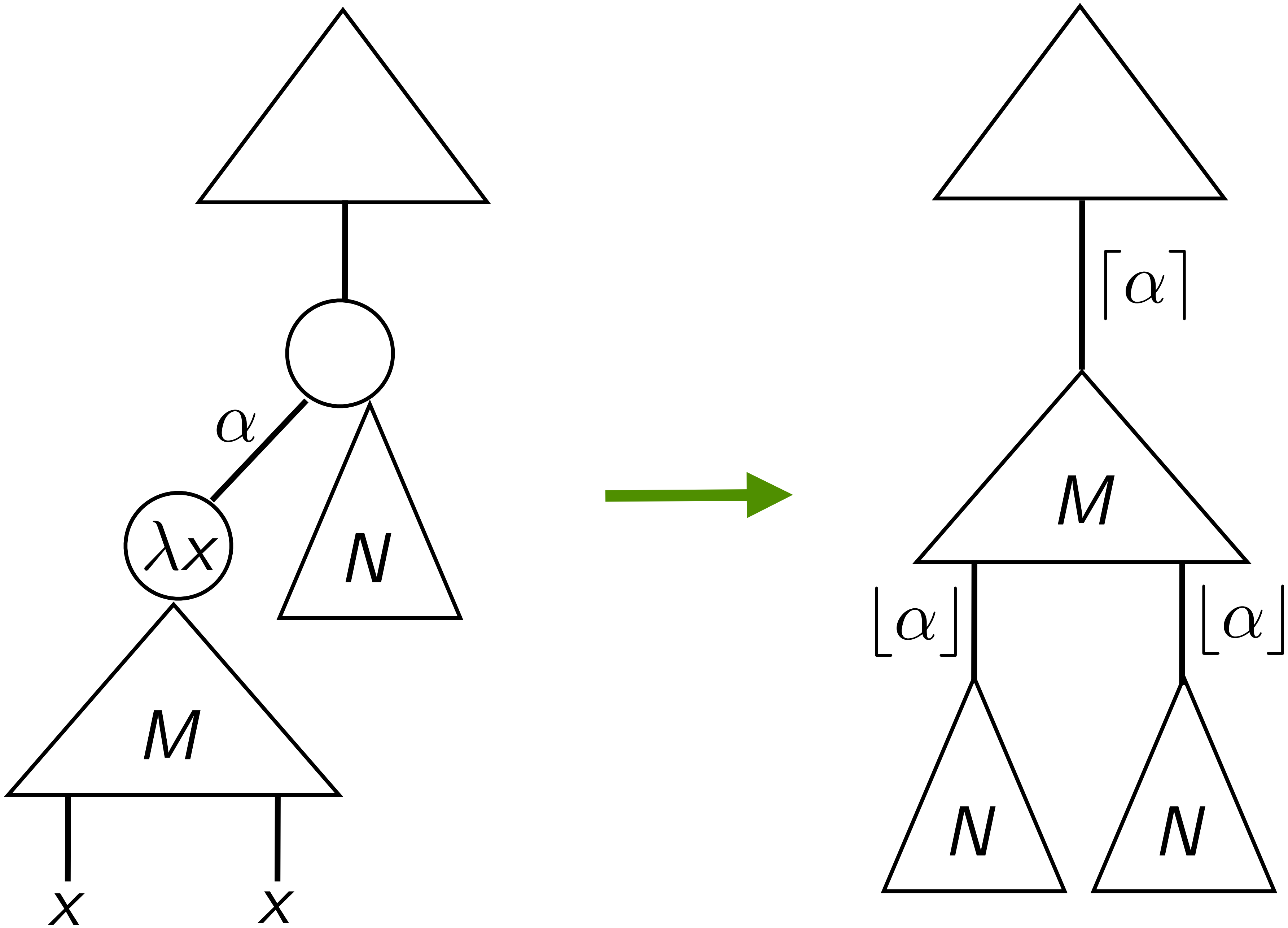
$$(\lambda x.M)^\alpha N \longrightarrow M^{\lceil \alpha \rceil}\{x := N^{\lfloor \alpha \rfloor}\}$$

α is the **name** of that redex

where

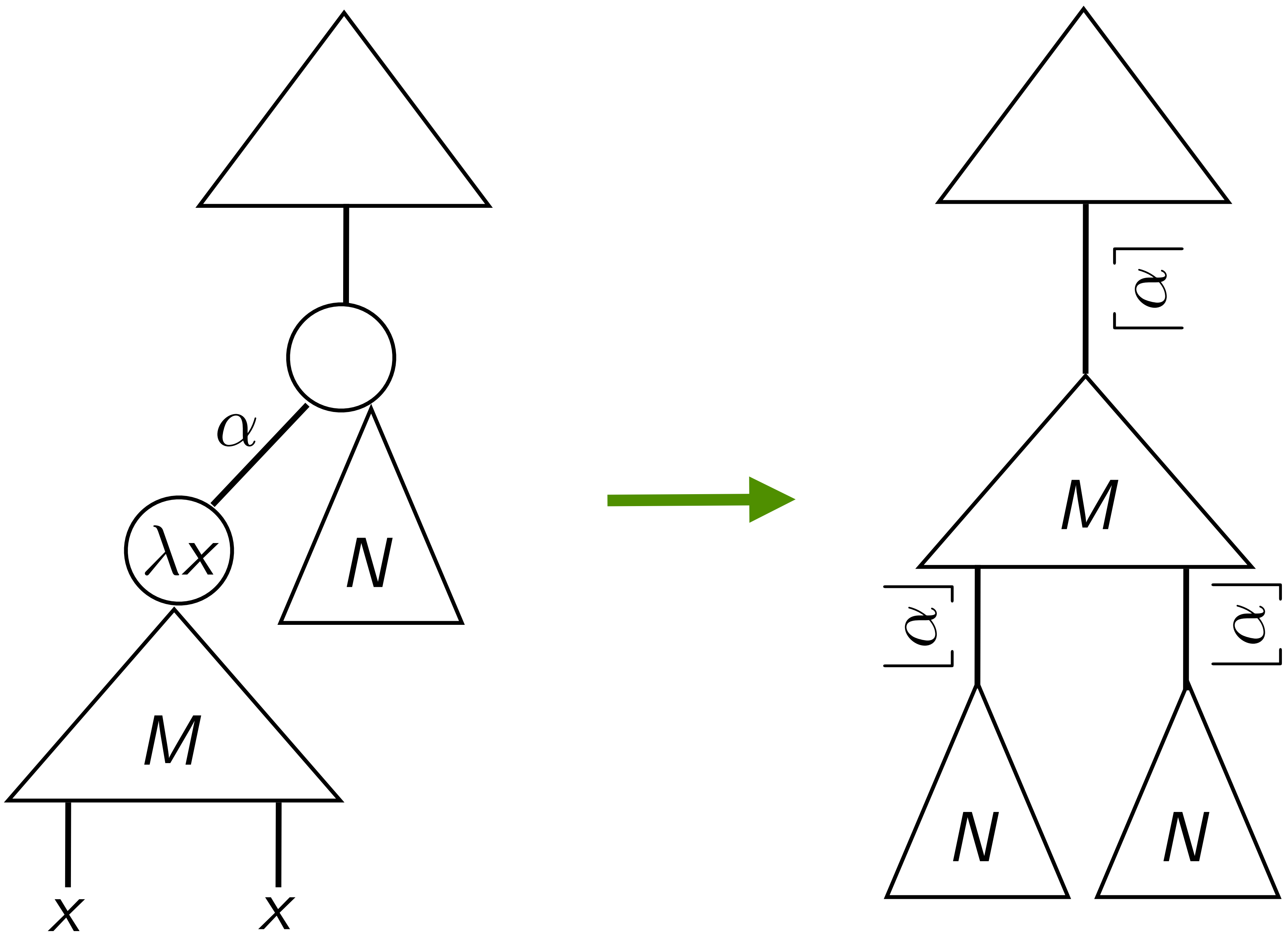
$$(M^\alpha)^\beta = M^{\alpha\beta} \quad \text{and} \quad (M^\alpha)\{x := N\} = (M\{x := N\})^\alpha$$

A labeled lambda-calculus (2/3)



abstract syntax trees of labeled λ-terms

A labeled lambda-calculus (2/3)



A labeled lambda-calculus (3/3)

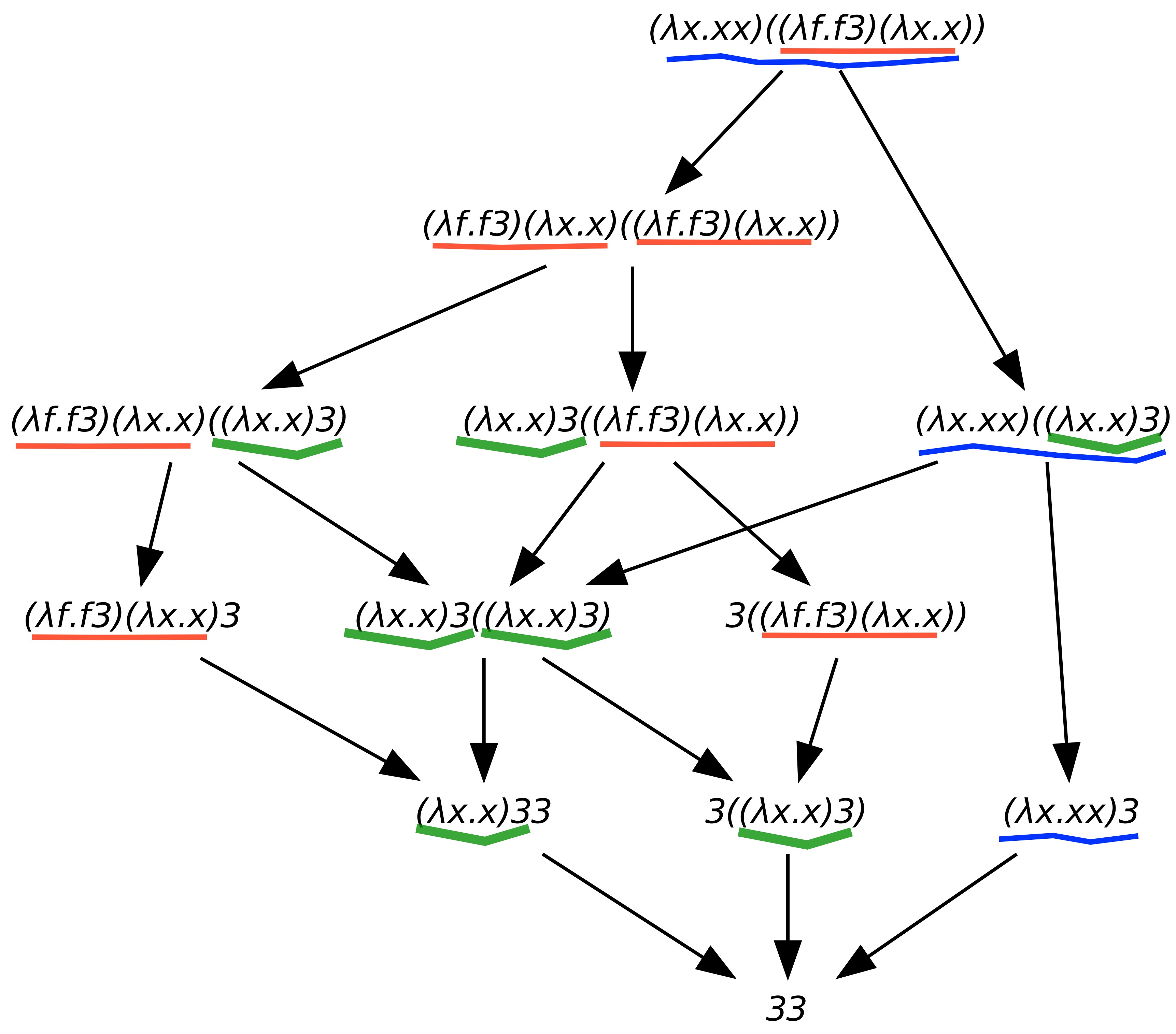
- Labels are strings of atomic labels:

$$\alpha, \beta, \ldots \quad ::= \quad \underbrace{\mathsf{a}, \mathsf{b}, \mathsf{c}, \ldots \mid \lceil \alpha \rceil \mid \lfloor \alpha \rfloor}_{\texttt{atomic labels}} \mid \alpha\beta \mid \epsilon$$

- Labels are strings of atomic labels:

$\mathsf{a}, \mathsf{b}, \mathsf{c}, \ldots$	atomic letters
$\lceil \alpha \rceil, \lfloor \alpha \rfloor, \ldots$	overlined, underlined labels
$\alpha\beta$	compound labels
$\epsilon = \lfloor \epsilon \rfloor = \lceil \epsilon \rceil$	empty label

Example



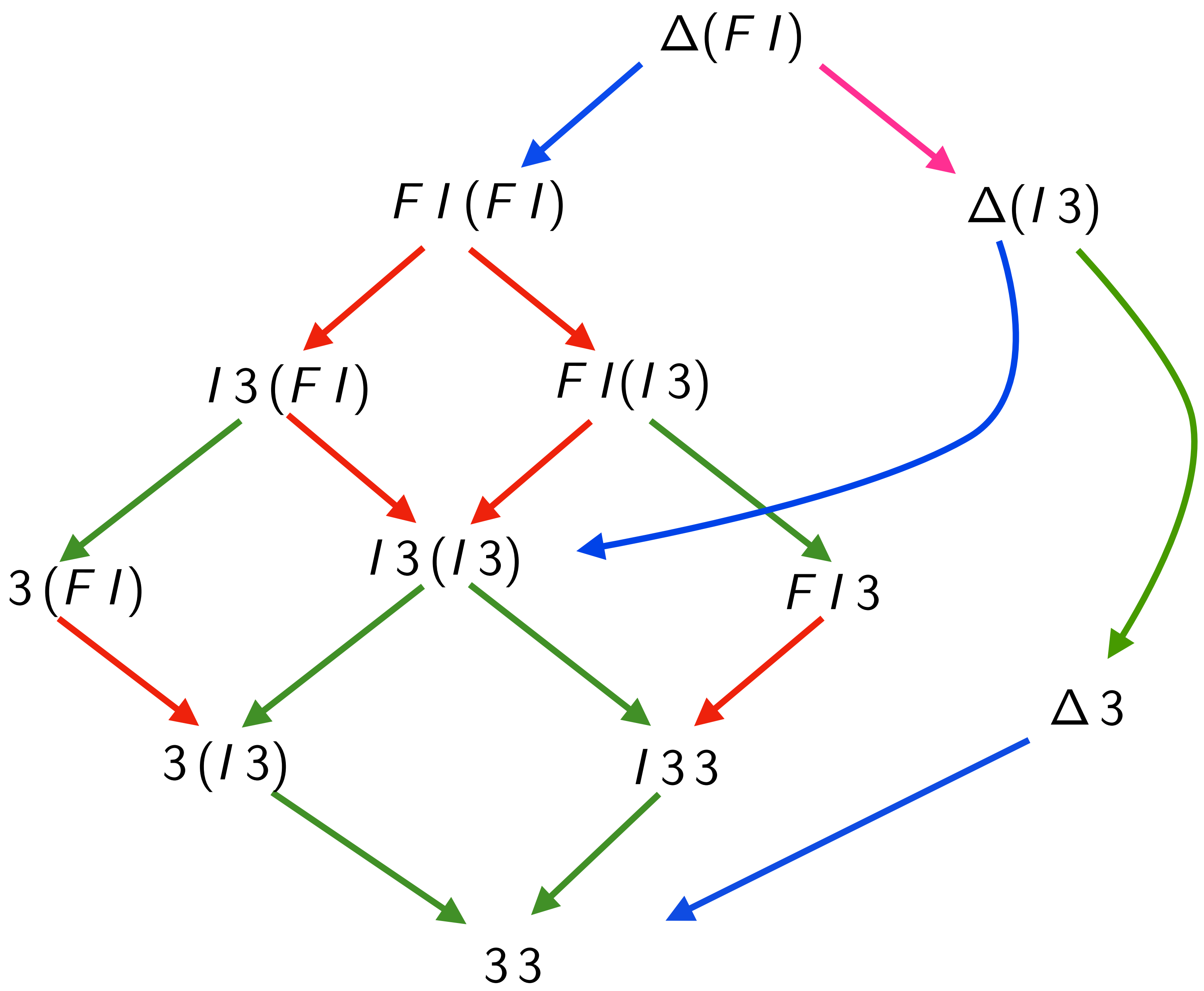
- 3 redex families: **red**, **blue**, **green**.

Example

$\Delta = \lambda x.x\,x$

$F = \lambda f.f\,3$

$I = \lambda x.x$



Example

$$\Delta = \lambda x.(x^c\, x^d)^b$$

$$F = \lambda f.(f^k\, 3^\ell)^j$$

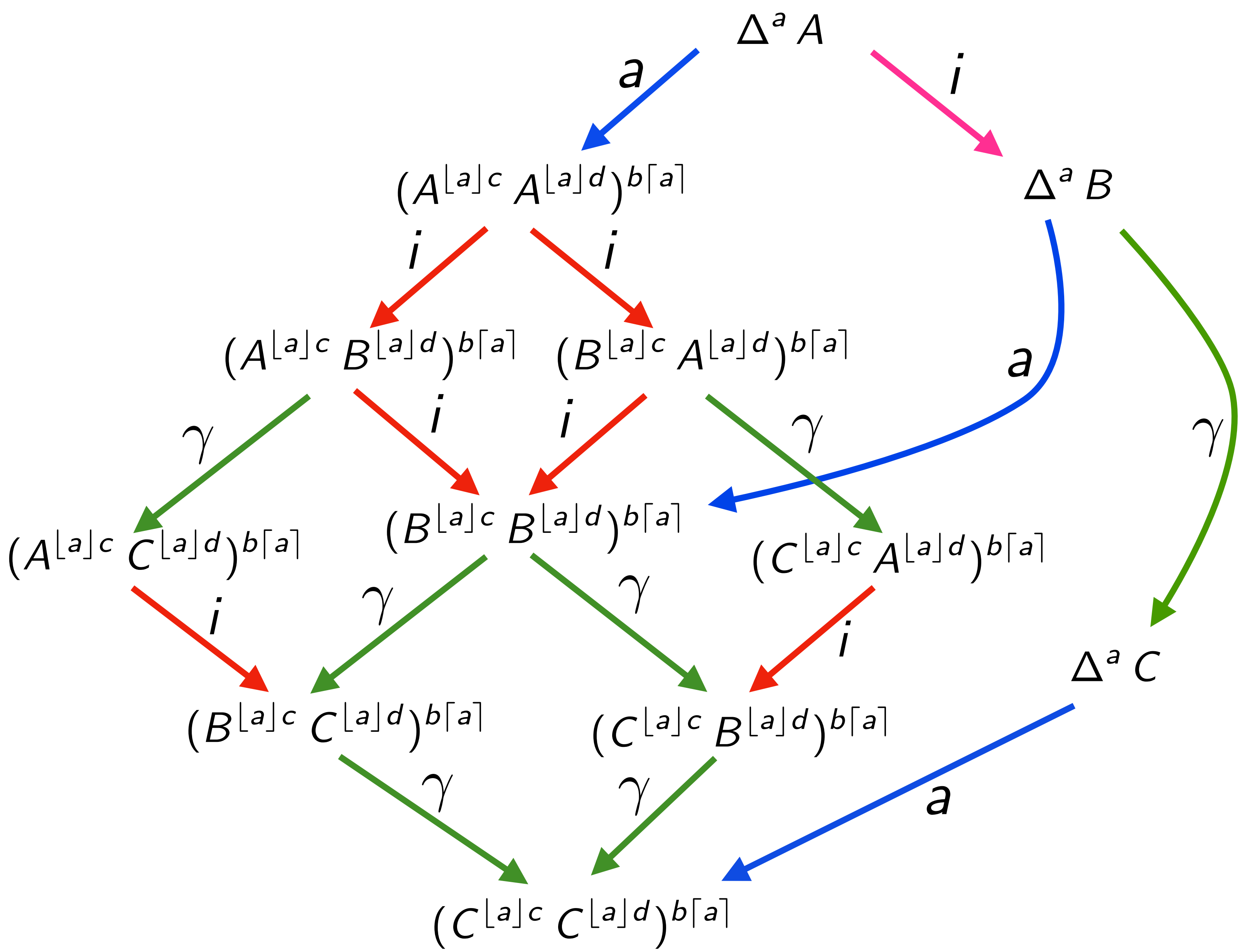
$$I = \lambda x.x^v$$

$$A = (F^i\, I^u)^q$$

$$B = (I^\gamma\, 3^\ell)^q$$

$$C = 3^{\ell\lfloor\gamma\rfloor v\lceil\gamma\rceil q}$$

$$\gamma = u\lfloor i\rfloor k$$



3 redexes names: $a, i, \gamma = u\lfloor i\rfloor k$

Example

$$D = \lambda x.(x^c\, x^d)^b$$
$$\Delta = \lambda x.(x^g\, x^h)^f$$

$$\Omega = D^a\, \Delta^e$$
$$\Big\downarrow a$$
$$\Omega_1 = (\Delta^{\gamma_1}\ \Delta^{\delta_1})^{b\lceil a\rceil}$$
$$\Big\downarrow \gamma_1$$
$$\Omega_2 = (\Delta^{\gamma_2}\ \Delta^{\delta_2})^{f\lceil\gamma_1\rceil b\lceil a\rceil}$$
$$\Big\downarrow \gamma_2$$
$$\Omega_3 = (\Delta^{\gamma_3}\ \Delta^{\delta_3})^{f\lceil\gamma_2\rceil f\lceil\gamma_1\rceil b\lceil a\rceil}$$
$$\Big\downarrow \gamma_3$$
$$\Omega_4 = (\Delta^{\gamma_4}\ \Delta^{\delta_4})^{f\lceil\gamma_3\rceil f\lceil\gamma_2\rceil f\lceil\gamma_1\rceil b\lceil a\rceil}$$
$$\Big\downarrow \gamma_4$$

$$\gamma_1 = e\lfloor a\rfloor c$$
$$\gamma_2 = \delta_1\lfloor\gamma_1\rfloor g$$
$$\gamma_3 = \delta_2\lfloor\gamma_2\rfloor g$$
$$\gamma_4 = \delta_3\lfloor\gamma_3\rfloor g$$

$$\delta_1 = e\lfloor a\rfloor d$$
$$\delta_2 = \delta_1\lfloor\gamma_1\rfloor h$$
$$\delta_3 = \delta_2\lfloor\gamma_2\rfloor h$$
$$\delta_4 = \delta_2\lfloor\gamma_2\rfloor h$$

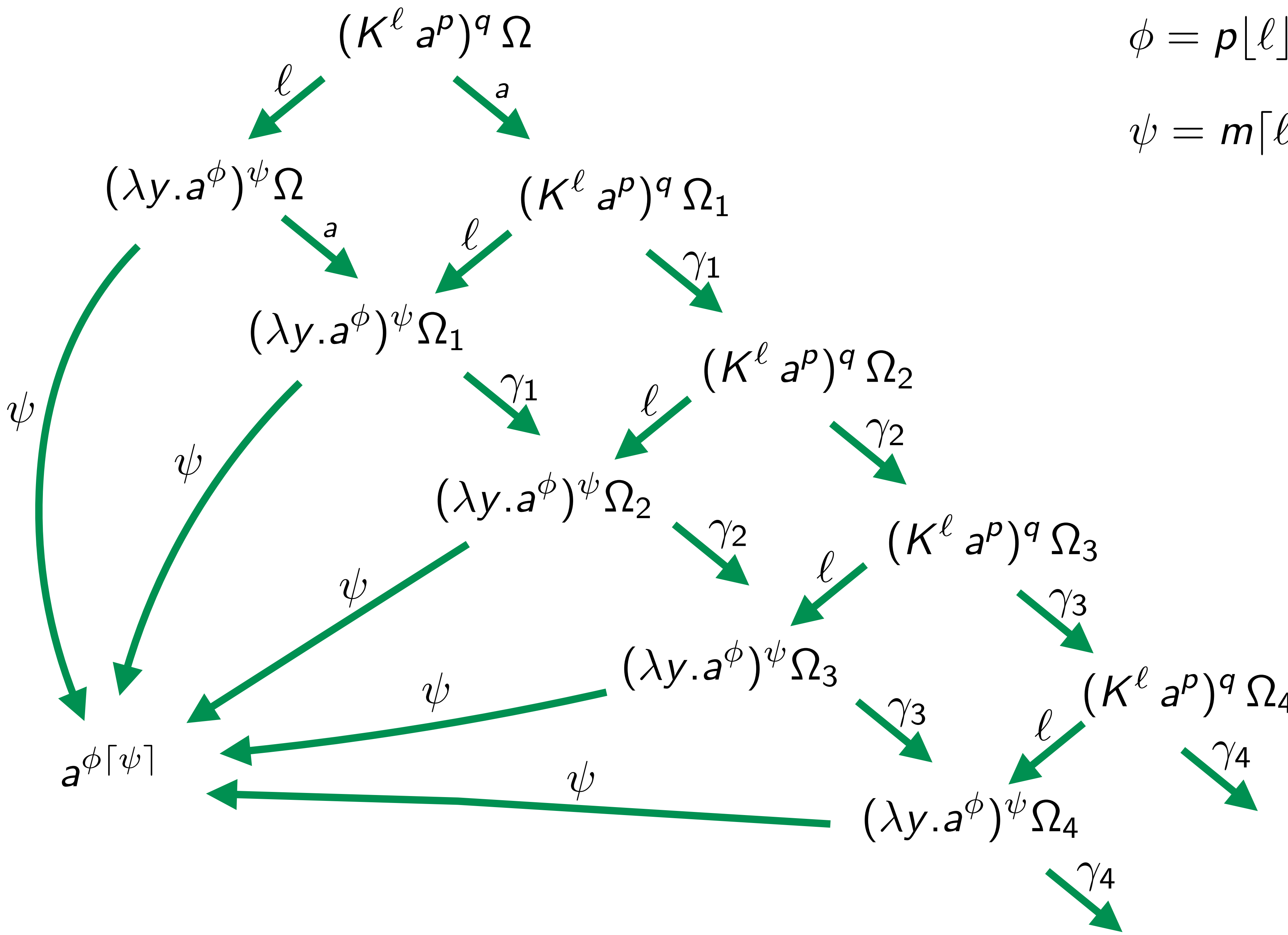
redexes names: $a, \gamma_1, \gamma_2, \gamma_3, \ldots$

Example

$$K = \lambda x.(\lambda y.x^n)^m$$

$$\phi = p\lfloor \ell \rfloor n$$

$$\psi = m\lceil \ell \rceil q$$



redexes names: $\ell, \psi, a, \gamma_1, \gamma_2, \gamma_3, \ldots$

Example

$$F = \lambda f.(f^c\, 3^d)^b$$
$$I = \lambda x.x^v$$
$$\Delta = \lambda x.(x^k\, x^\ell)^j$$

$$\begin{tikzcd}[column sep=small]
 & F^a(I^u\,\Delta^i)^q \arrow[dl, "a"'] \arrow[dr, "u"] & \\
((I^u\,\Delta^i)^{q\lfloor a\rfloor c}\, 3^d)^{b\lceil a\rceil} \arrow[dr, "u"'] & & F^a\,\Delta^{i\lfloor u\rfloor v\lceil u\rceil q} \arrow[dl, "a"] \\
 & (\Delta^{i\lfloor u\rfloor v\lceil u\rceil q\lfloor a\rfloor c}\, 3^d)^{b\lceil a\rceil} &
\end{tikzcd}$$

2 independent redexes a and u creates the new one $i\lfloor u\rfloor v\lceil u\rceil q\lfloor a\rfloor c$

Empirical facts (bis)

- **deterministic** result when it exists — **Church-Rosser**
- multiple reduction strategies
- **terminating** strategy ?
- **efficient** reduction strategy ? — **optimal reduction**
- **worst** reduction strategy ?
- when all reductions are finite ? — **strong normalisation**
- when finite, the reduction graph has a **lattice** structure ? — **YES!**

Permutation equivalence (1/7)

- **Proposition** `[residuals of labeled redexes]`

  $S \in R/\rho$ implies $\mathrm{name}(R) = \mathrm{name}(S)$

- **Definition** `[created redexes]` Let $\rho : M \xrightarrow{\star} N$

  we say that ρ **creates** R in M when $\nexists R',\ R \in R'/\rho$.

- **Proposition** `[created labeled redexes]`

  If S creates R, then $\mathrm{name}(S)$ is strictly contained in $\mathrm{name}(R)$.

Permutation equivalence (2/7)

Proof (cont'd) Created redexes contains names of creator

$$\underbrace{(\lambda x.\cdots(x^\beta\, N)\cdots)}_{\alpha}{}^{\alpha}\,(\lambda y.M)^\gamma \rightarrow \cdots\underbrace{((\lambda y.M)^{\gamma\lfloor\alpha\rfloor\beta}\, N')}_{\gamma\lfloor\alpha\rfloor\beta}\cdots$$

$$\alpha \xrightarrow{\text{creates}} \gamma\lfloor\alpha\rfloor\beta$$

$$\underbrace{((\lambda x.(\lambda y.M)^\gamma)^\alpha}_{\alpha} N)^\beta\, P \rightarrow \underbrace{(\lambda y.M')^{\gamma\lceil\alpha\rceil\beta}\, P}_{\gamma\lceil\alpha\rceil\beta}$$

$$\alpha \xrightarrow{\text{creates}} \gamma\lceil\alpha\rceil\beta$$

$$\underbrace{((\lambda x.\,x^\gamma)^\alpha\,(\lambda y.M)^\delta)^\beta}_{\alpha}\, N \rightarrow \underbrace{(\lambda y.M)^{\delta\lfloor\alpha\rfloor\gamma\lceil\alpha\rceil\beta}\, N}_{\delta\lfloor\alpha\rfloor\gamma\lceil\alpha\rceil\beta}$$

$$\alpha \xrightarrow{\text{creates}} \delta\lfloor\alpha\rfloor\gamma\lceil\alpha\rceil\beta$$

Permutation equivalence (3/7)

- **Labeled laws** $M^\alpha\{x := N\} = (M\{x := N\})^\alpha \qquad (M^\alpha)^\beta = M^{\alpha\beta}$

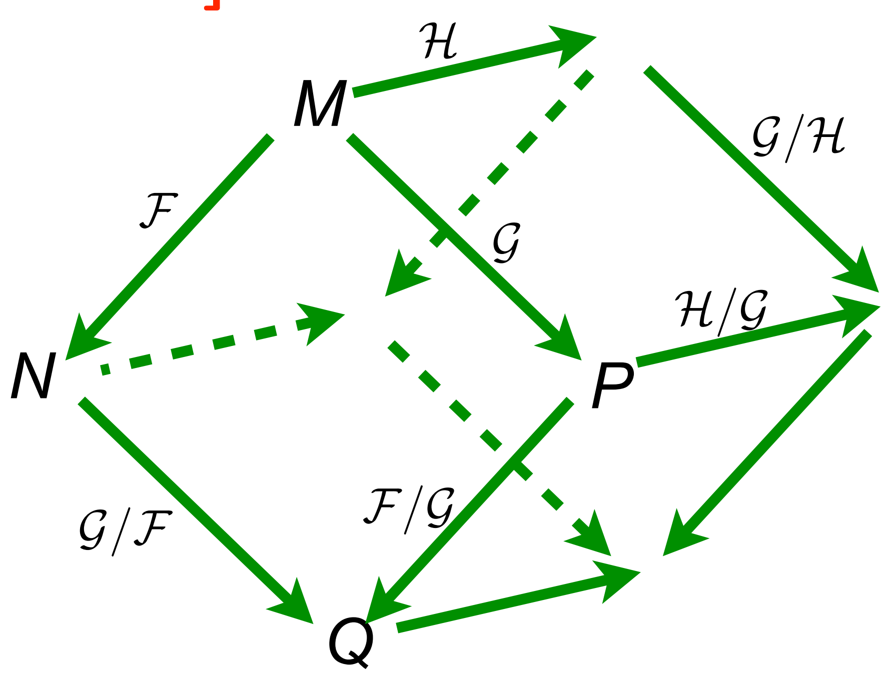
  If $M \longrightarrow N$, then $M^\alpha \longrightarrow N^\alpha$

- **Labeled parallel moves lemma+** [74]

  If $M \xrightarrow{\mathcal{F}} N$ and $M \xrightarrow{\mathcal{G}} P$, then $N \xrightarrow{\mathcal{G}/\mathcal{F}} Q$ and $P \xrightarrow{\mathcal{F}/\mathcal{G}} Q$ for some Q.

- **Parallel moves lemma++** [The Cube Lemma]

  still holds.



Permutation equivalence (4/7)

- Labels do not break Church-Rosser, nor residuals

- Labels refine λ-calculus:
  - any unlabeled reduction can be performed in the labeled calculus
  - but two cofinal unlabeled reductions may no longer be cofinal

  Take $I(I3)$ with $I = \lambda x.x$.

$$
\begin{tikzcd}
 & I^b(J^e 3^g)^d \arrow[dl] \arrow[dr] & \\
(J^e\,3^g)^{d\lfloor b\rfloor c\lceil b\rceil} & & I^b 3^{g\lfloor e\rfloor f\lceil e\rceil d}
\end{tikzcd}
$$

$$
\begin{aligned}
I &= \lambda x.x^c \\
J &= \lambda x.x^f
\end{aligned}
$$

Permutation equivalence (5/7)

- **Definition** `[pure labeled calculus]`

  Pure labeled terms are labeled terms where all subterms have non empty labels.

- **Theorem** `[labeled permutation equivalence, 76]`

  Let ρ and σ be coinitial pure labeled reductions.
  Then $\rho \simeq \sigma$ iff ρ and σ are labeled cofinal.

  Proof Let $\rho \simeq \sigma$. Then obvious because of labeled parallel moves lemma.
  Conversely, we apply standardization thm and following lemma.

Permutation equivalence (6/7)

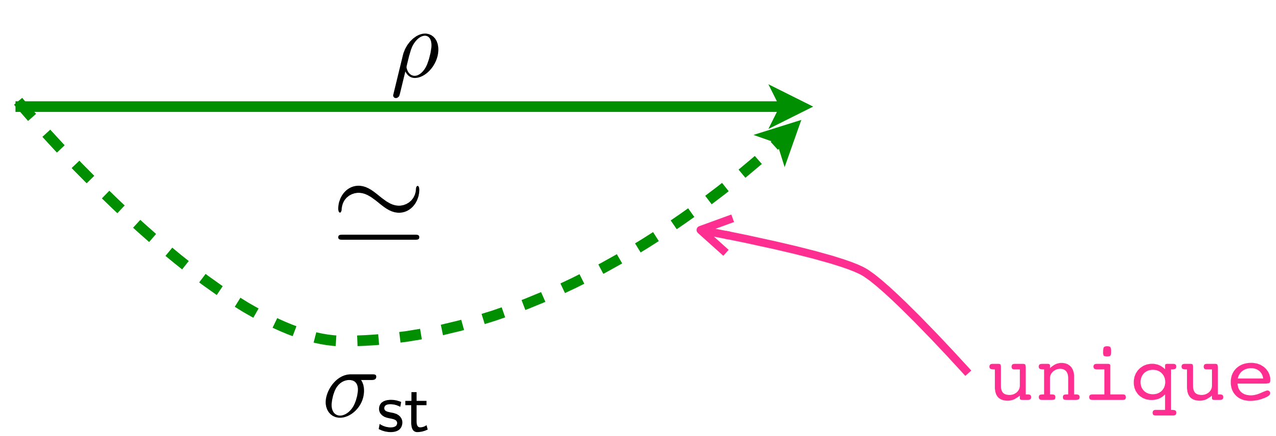
- **Definition:** The following reduction is **standard**

$$\rho : M = M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} M_2 \cdots \xrightarrow{R_n} M_n = N$$

iff for all i and j, $i < j$, then R_j is not residual along ρ of some R'_j to the left of R_i in M_{i-1}.

- **Standardization** [Curry 50] Let $M \xrightarrow{\star} N$. Then $M \xrightarrow[\text{st}]{\star} N$.

- **Labeled standardization** $\forall \rho,\ \exists!\, \sigma_{\text{st}},\ \rho \simeq \sigma_{\text{st}}$



Permutation equivalence (7/7)

- **Notation** `[prefix ordering]` $\rho \sqsubseteq \sigma$ for $\exists \tau .\, \rho\tau \simeq \sigma$

- **Corollary** `[labeled prefix ordering]`

  Let $\rho : M \xrightarrow{\star} N$ and $\sigma : M \xrightarrow{\star} P$ be coinitial pure labeled reductions. Then $\rho \sqsubseteq \sigma$ iff $N \xrightarrow{\star} P$.

- **Corollary** `[lattice of labeled reductions]`

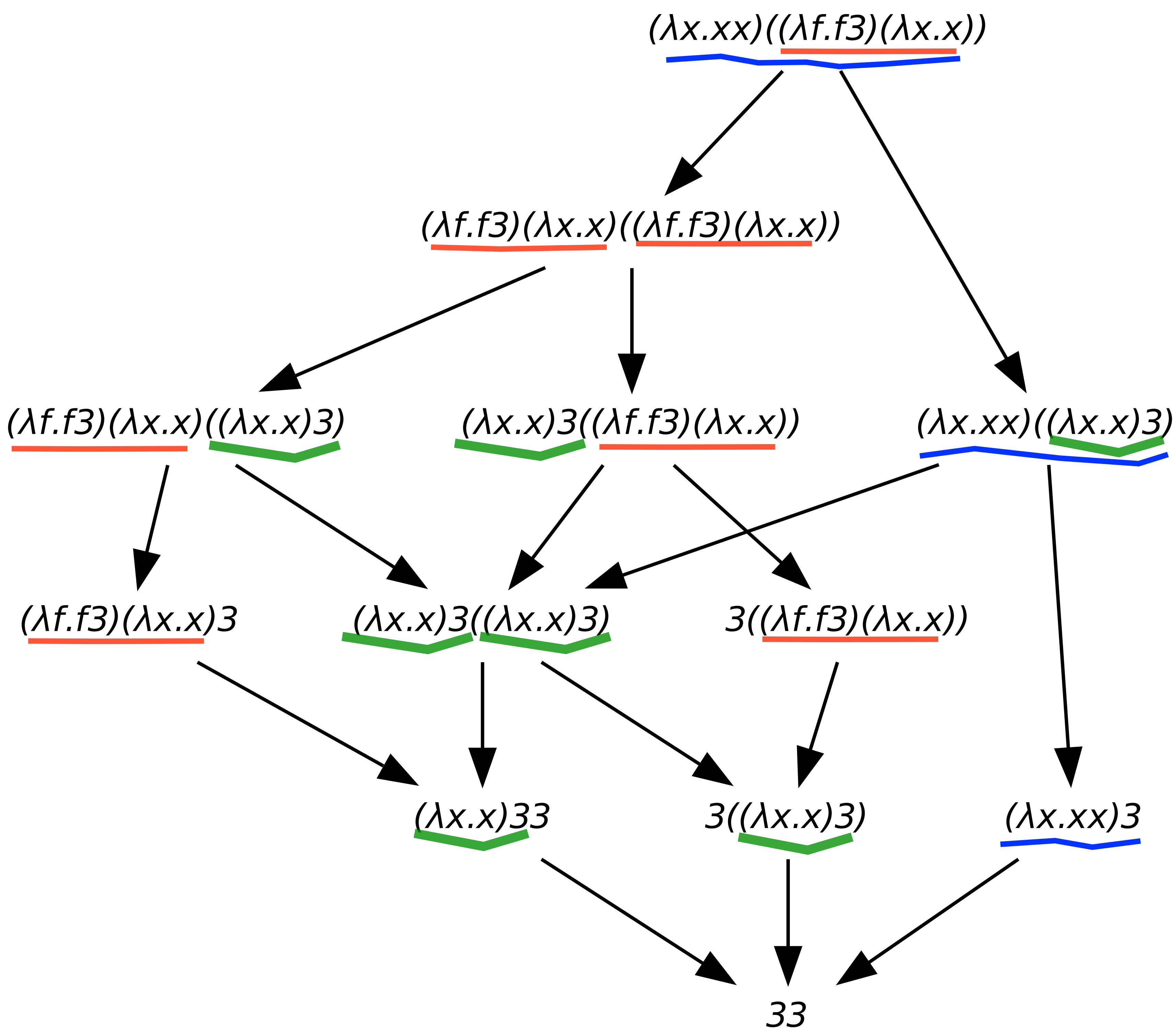
  Labeled reduction graphs are upwards semi lattices for any pure labeling.

In other terms, reductions up-to permutation equivalence is a push-out category.

Exercise Try on $(\lambda x.x)((\lambda y.(\lambda x.x)a)b)$ or $(\lambda x.xx)(\lambda x.xx)$

Redex families

Example



- 3 redex families: **red**, **blue**, **green**.

hRedexes

- **Definition** `[hRedex]`

  hRedex is a pair $\langle \rho, R \rangle$ where R is a redex in final term of ρ

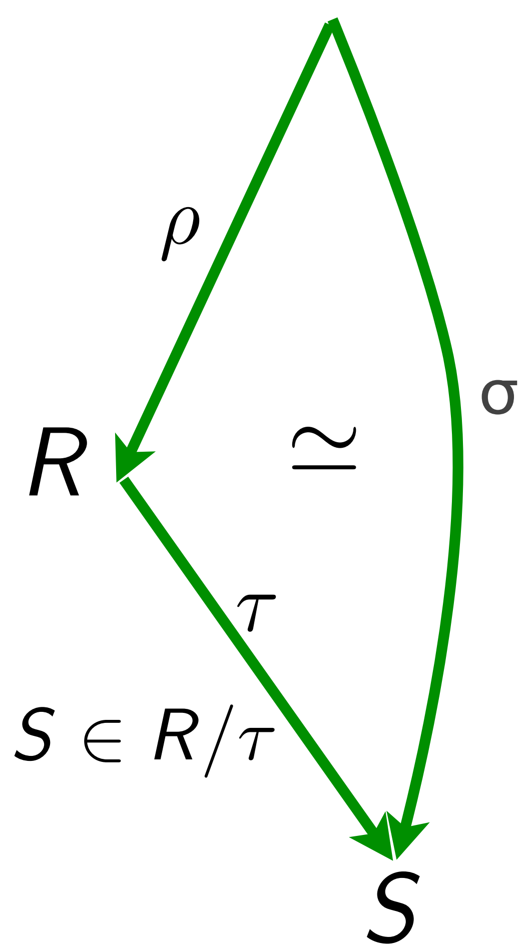
- **Definition** `[copies of hRedex]`

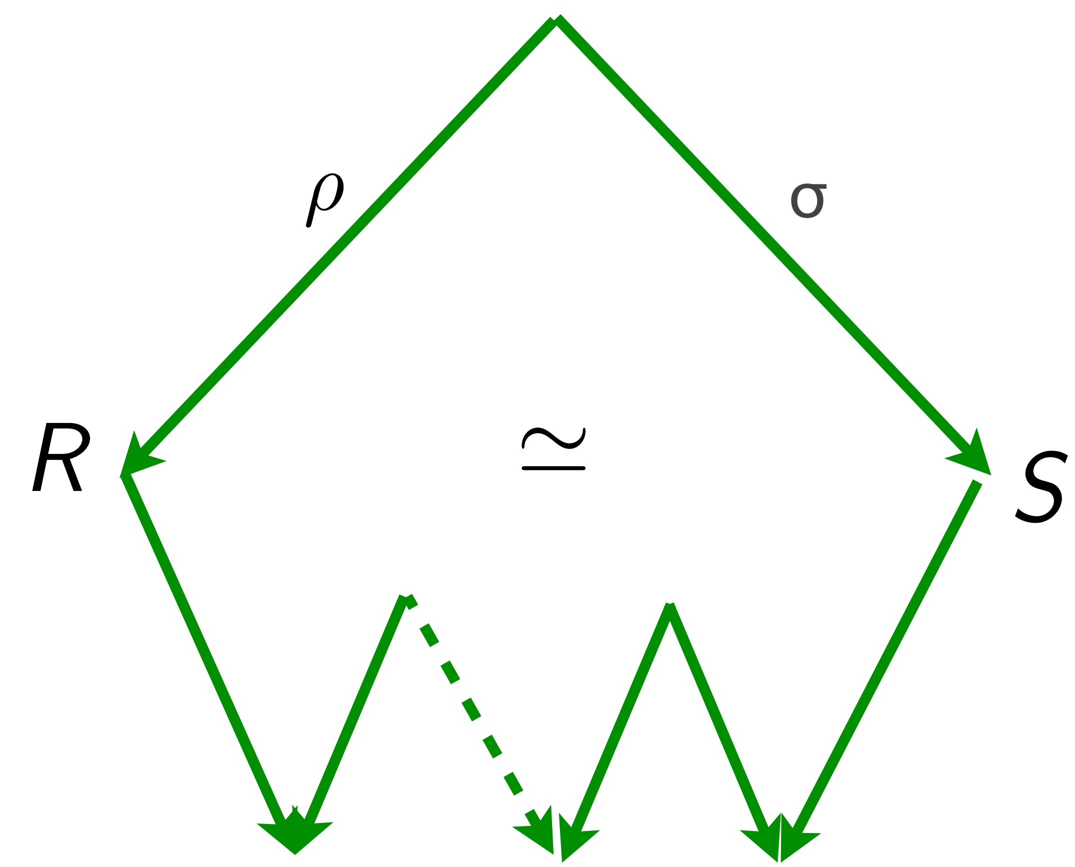
  $\langle \rho, R \rangle \leq \langle \sigma, S \rangle$ when $\exists \tau.\, \rho\tau \simeq \sigma$ and $S \in R/\tau$

- **Definition** `[families of hRedexes]`

  $\langle \rho, R \rangle \sim \langle \sigma, S \rangle$ for reflexive, symmetric, transitive closure of the copy relation.

Labels and history (1/4)





$\langle \rho, R \rangle \leq \langle \sigma, S \rangle$

$\langle \rho, R \rangle \sim \langle \sigma, S \rangle$

$$\mathsf{name}(R) = \mathsf{name}(S)$$

Labels and history (2/4)

- **Proposition** [same history → same name]

  In the labeled λ-calculus, for any labeling, we have:

  $$\langle \rho, R \rangle \sim \langle \sigma, S \rangle \text{ implies } \mathsf{name}(R) = \mathsf{name}(S)$$

- The opposite direction is clearly not true for any labeling

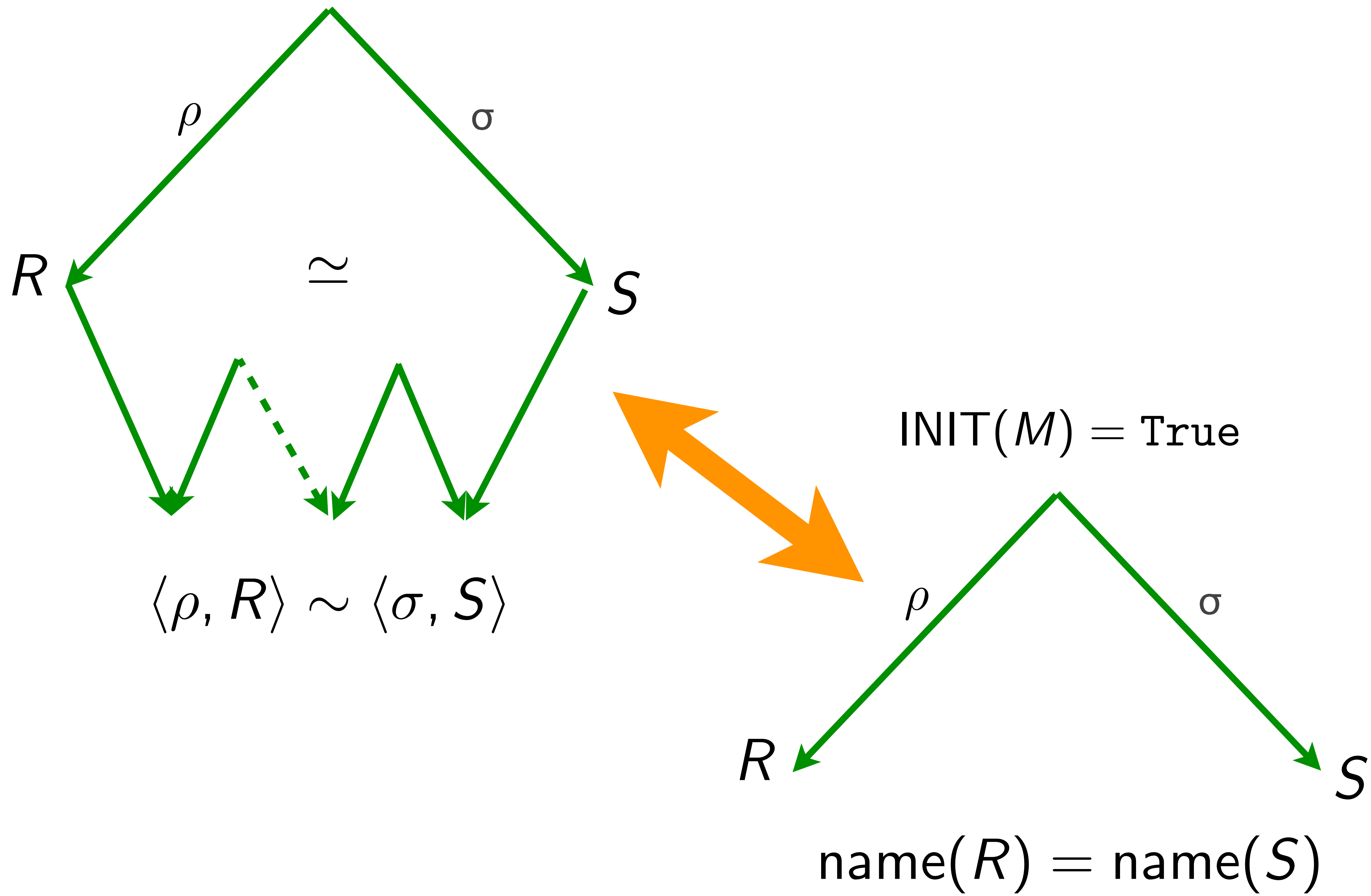
  (For instance, take all labels equal)

- But it is true when all labels are distinct atomic letters in the initial term.

- **Definition** [all labels distinct letters]

  $\mathsf{INIT}(M) = \mathtt{True}$ when all labels in M are distinct letters.

Labels and history (3/4)



Labels and history (4/4)

- **Theorem** `[same history = same name, 76]`

  When $\mathrm{INIT}(M)$ and reductions ρ and σ start from M :

  $$\langle \rho, R \rangle \sim \langle \sigma, S \rangle \quad \text{iff} \quad \mathsf{name}(R) = \mathsf{name}(S)$$

- **Corollary** `[decidability of family relation]`

  The family relation is decidable (although complexity is proportional to length of standard reduction).

Finite developments

Parallel steps revisited (1/3)

- parallel steps were defined with inside-out strategy

  `[à la Martin-Löf]`

  Can we take any order as a reduction strategy ?

- **Definition** A **reduction relative** to a set $\mathcal{F}$ of redexes in M is any reduction contracting only residuals of $\mathcal{F}$.

  A **development** of $\mathcal{F}$ is any maximal relative reduction of $\mathcal{F}$.

Parallel steps revisited (2/3)

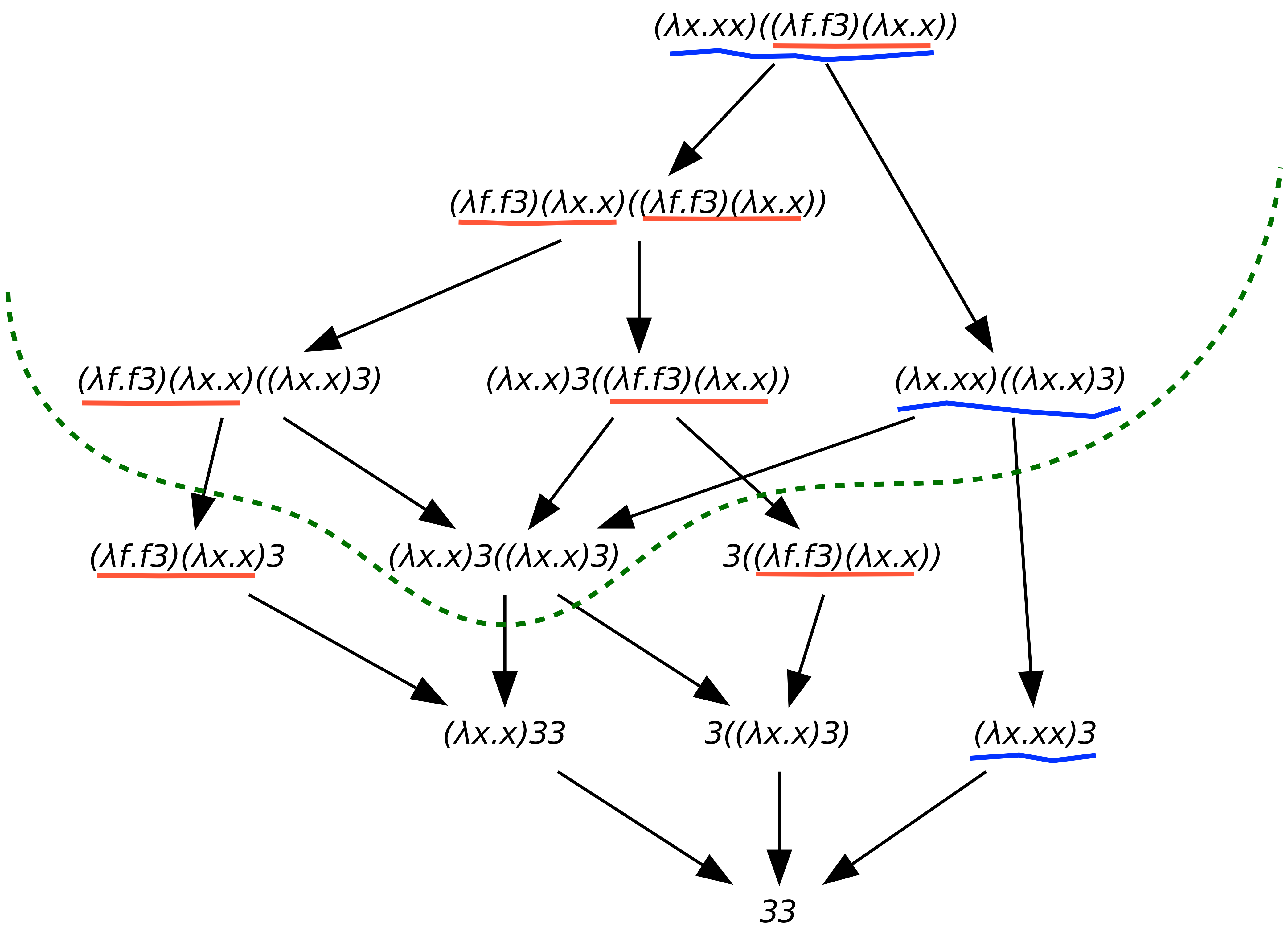
- **Theorem** `[Finite Developments, Curry, 50]`

  Let $\mathcal{F}$ be set of redexes in M.

  (1) there are no infinite relative reductions of $\mathcal{F}$,
  (2) they all finish on same term N
  (3) Let R be redex in M. Residuals of R by all finite developments of $\mathcal{F}$ are the same.

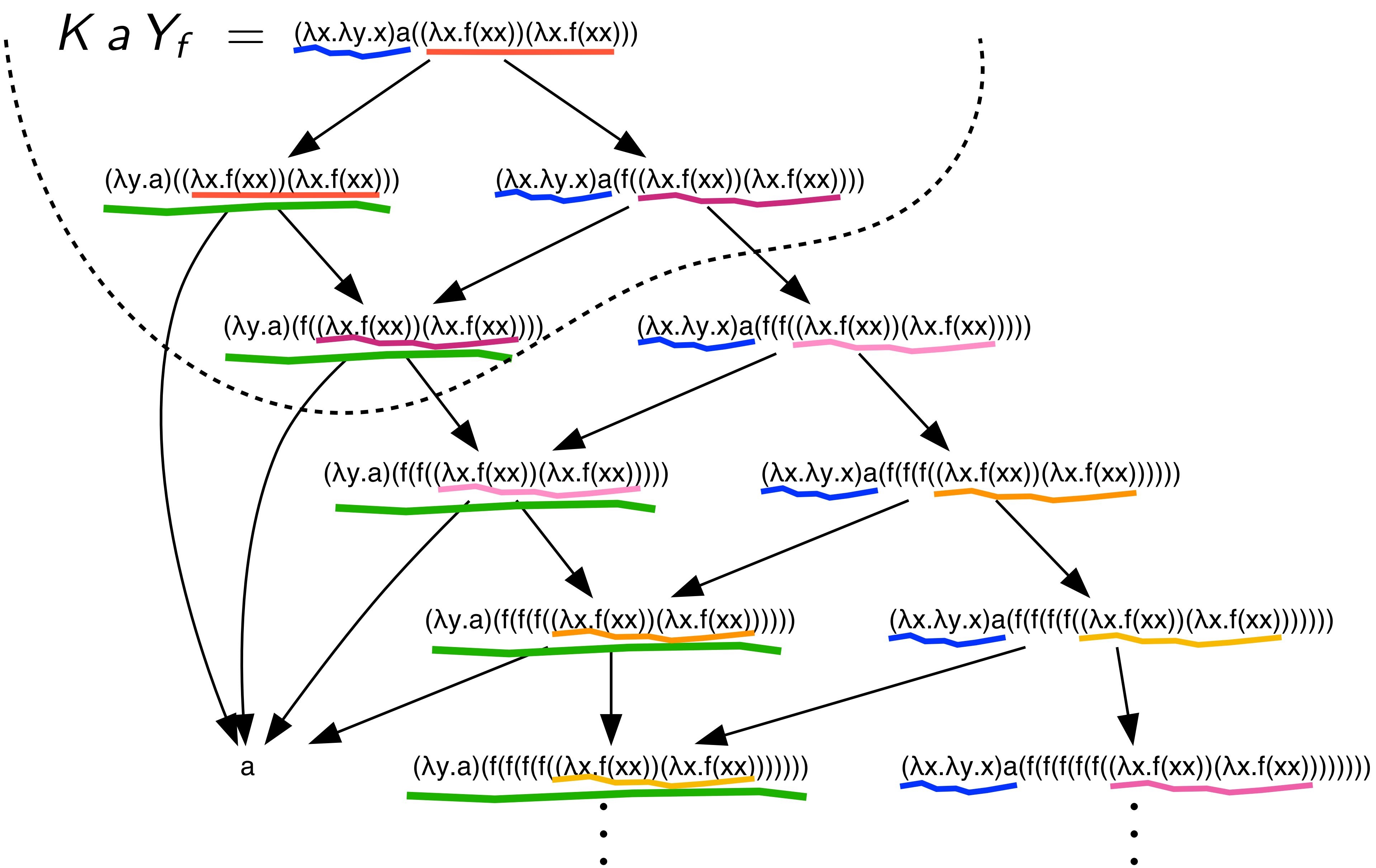
- Similar to the parallel moves lemma, but we considered a particular inside-out reduction strategy.

Example



developments of **red**, **blue**.

Example



developments of **red**, **blue**.

Parallel steps revisited (3/3)

- **Notation** `[parallel reduction steps]`

  Let $\mathcal{F}$ be set of redexes in M. We write $M \xrightarrow{\mathcal{F}} N$

  if a development of $\mathcal{F}$ connects M to N.

- This notation is consistent with previous definition

  (since inside-out parallel step is a particular development)

- Corollaries of FD thm are also parallel moves + cube lemmas

Finite and infinite reductions (1/3)

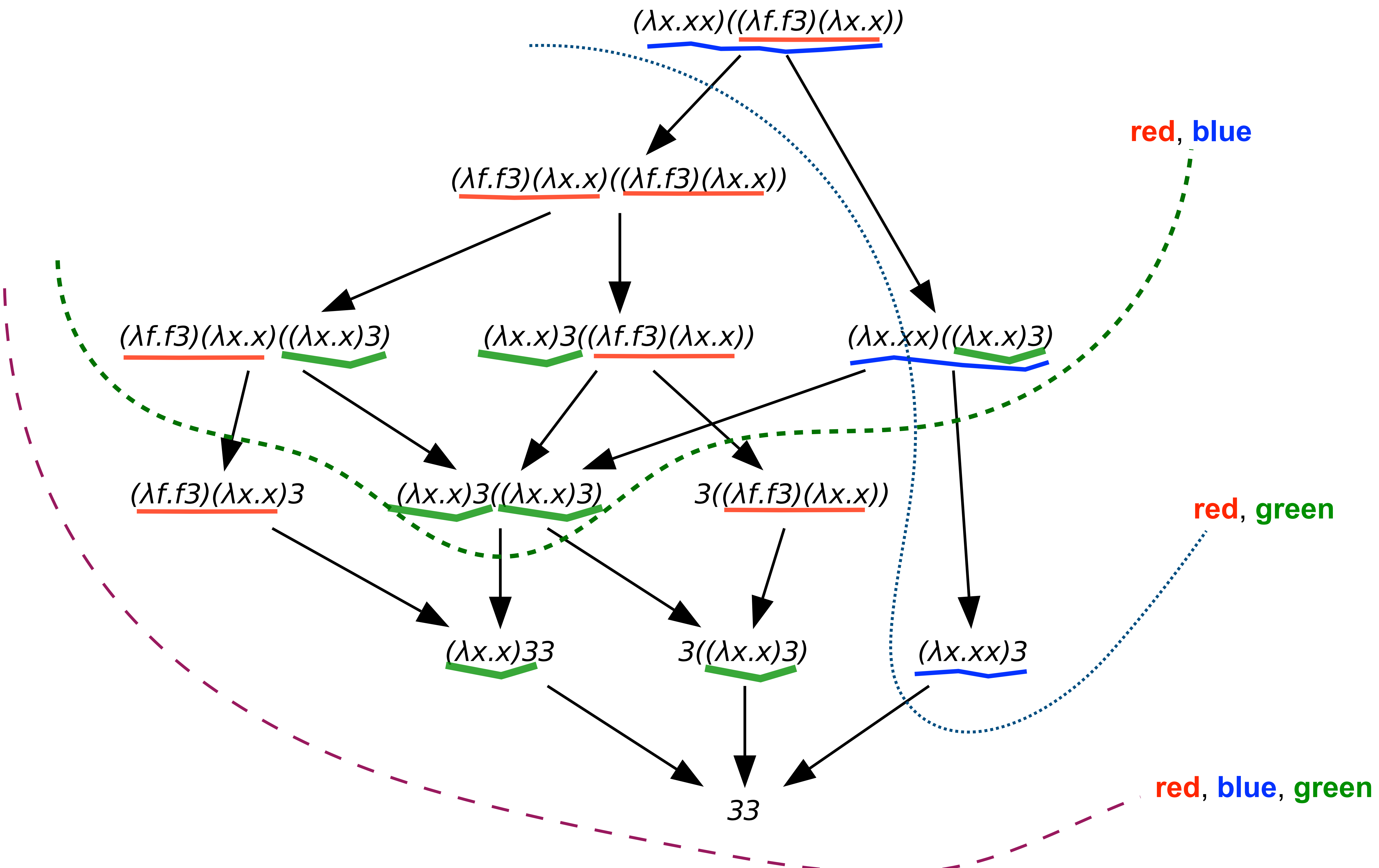
- **Definition** A **reduction relative** to a set $\mathcal{F}$ of redex families is any reduction contracting redexes in families of $\mathcal{F}$.

  A **development** of $\mathcal{F}$ is any maximal relative reduction.

- **Theorem** `[Generalized Finite Developments+, 76]`
  Let $\mathcal{F}$ be a finite set of redex families.

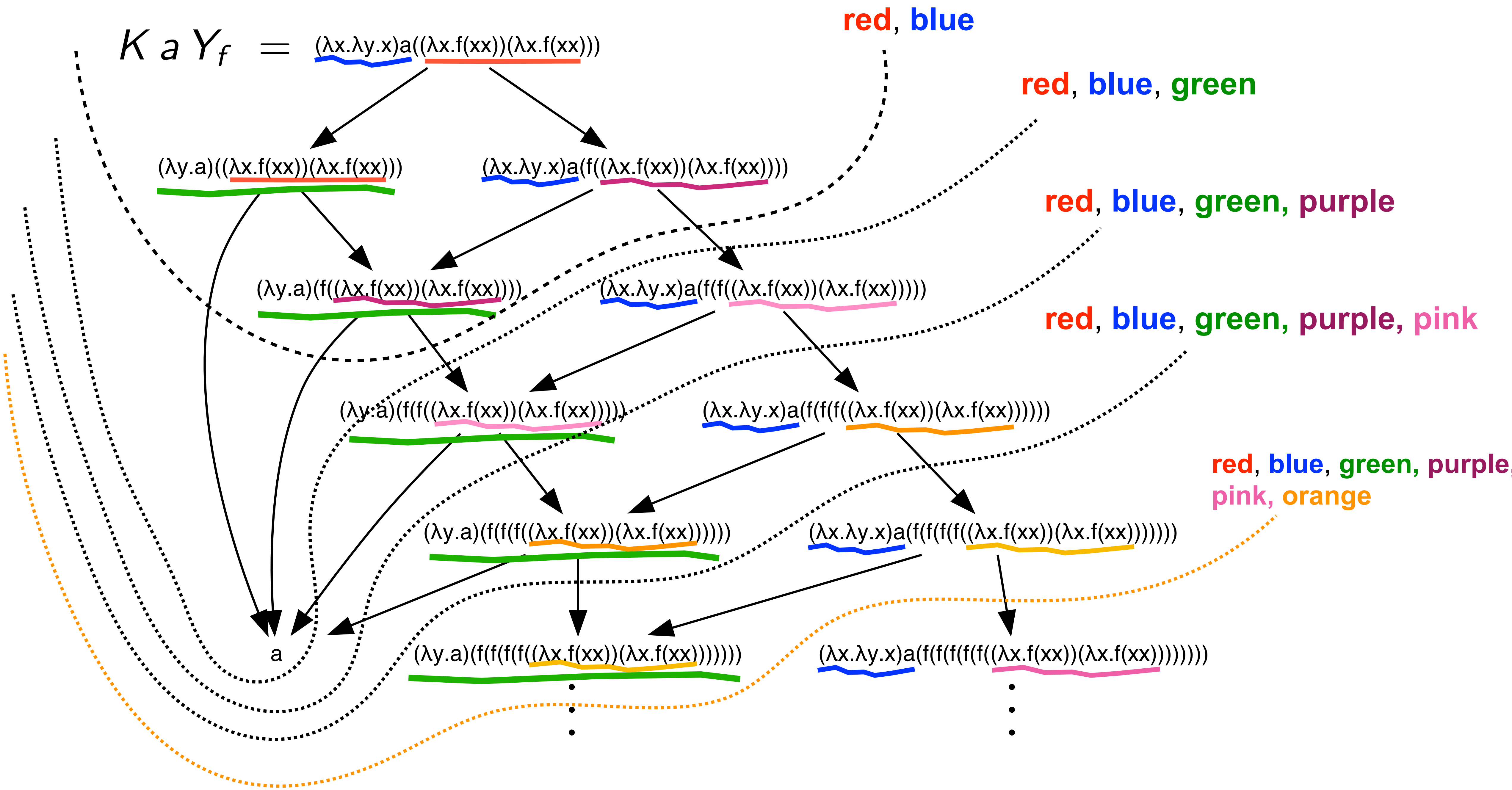
  (1) there are no infinite reductions relative to $\mathcal{F}$,
  (2) they all finish on same term N
  (3) All developments are equivalent by permutations.

Example



- 3 redex families: **red**, **blue**, **green**.

Example



developments of families.

Finite and infinite reductions (2/3)

- **Corollary** An **infinite reduction** contracts an **infinite set of redex families.**

- **Corollary** Any term generating a finite number of redex families strongly normalizes

finite number of redex families

strong normalization

Proof of the GFD thm

Bound on heights of labels

- **Definition** The height of a label is its nesting of underlines and overlines

$$h(a) = 0$$
$$h(\lceil\alpha\rceil) = h(\lfloor\alpha\rfloor) = 1 + h(\alpha)$$
$$h(\alpha\beta) = \max\{\alpha, \beta\}$$

- **Fact** Let $\mathcal{F}$ be a finite set of redex families, then there is an upper bound $H(\mathcal{F})$ on labels of subterms in reductions relative to $\mathcal{F}$.

When initial term is labeled with atomic letters, we have

$$H(\mathcal{F}) = \max\{h(\alpha) \mid \alpha \in \mathcal{F}\}$$

Proof of finite developments

- **Notation** $\tau(M^\alpha) = \alpha$ when M has an empty external label

- **Lemma 1** Let $M \xrightarrow{\star} M'$, then $h(\tau(M)) \leq h(\tau(M'))$

- **Lemma 2** Let $(\cdots((M\, M_1)^{\beta_1}\, M_2)^{\beta_2} \cdots M_n)^{\beta_n} \xrightarrow{\star} (\lambda x.N)^\alpha$

  Then $h(\tau(M)) \leq h(\alpha)$

- **Lemma 3** [Barendregt] Let $M\{x := N\} \xrightarrow{\star} (\lambda y.P)^\alpha$

  There are 2 cases:

  $M \xrightarrow{\star} (\lambda y.M')^\alpha$ and $M'\{x := N\} \xrightarrow{\star} P$

  $M \xrightarrow{\star} M' = (\cdots((x^\beta\, M_1)^{\beta_1}\, M_2)^{\beta_2} \cdots M_n)^{\beta_n}$ and $M'\{x := N\} \xrightarrow{\star} (\lambda y.P)^\alpha$

Proof of finite developments

- **Notation** Let $\mathcal{SN}_{\mathcal{F}}$ be the set of strongly normalizable terms w.r.t. reductions relative to $\mathcal{F}$.

- **Lemma** `[subst]` Let $\mathcal{F}$ be a finite set of redex families.
  $M, N \in \mathcal{SN}_{\mathcal{F}}$ implies $M\{x := N\} \in \mathcal{SN}_{\mathcal{F}}$

  Proof `[van Daalen]` by induction on $\langle H(\mathcal{F}) - h(\tau(N)), \mathrm{depth}(M), \|M\| \rangle$

- **Theorem** `GFD` Let $\mathcal{F}$ be a finite set of redex families.
  Then $M \in \mathcal{SN}_{\mathcal{F}}$ for all M.

  Proof by induction on $\|M\|$

Strong normalization

1st-order typed λ-calculus (1/2)

Residuals of redexes keep their types (of names)

Created redexes have lower types

$$\underbrace{(\lambda x.\cdots xN\cdots)}_{s\to t}\underbrace{(\lambda y.M)}_{s} \longrightarrow \cdots\underbrace{(\lambda y.M)}_{s}N'\cdots$$

creates

Finite number of redexes families

$$\underbrace{(\lambda x.\underbrace{\lambda y.M}_{t})}_{s\to t}NP \longrightarrow \underbrace{(\lambda y.M')}_{t}P$$

creates

Strong normalization

$$\underbrace{(\lambda x.x)}_{s\to s}\underbrace{(\lambda y.M)}_{s}N \longrightarrow \underbrace{(\lambda y.M)}_{s}N$$

creates

1st-order typed λ-calculus (2/2)

- **Typed λ-calculus** as a specific labeled calculus

$$s,\ t \ ::= \ \mathbb{N}, \mathbb{B} \ \mid \ s \rightarrow t$$

Decorate subterms with their types

$$(\lambda f.\,(f^{\mathbb{N}\rightarrow\mathbb{N}} 3^{\mathbb{N}})^{\mathbb{N}})^{(\mathbb{N}\rightarrow\mathbb{N})\rightarrow\mathbb{N}} I^{\mathbb{N}\rightarrow\mathbb{N}}$$

$$\rightarrow (I^{\mathbb{N}\rightarrow\mathbb{N}} 3^{\mathbb{N}})^{\mathbb{N}} \rightarrow 3^{\mathbb{N}}$$

Apply following rules to labeled λ-calculus

$$\lceil s \rightarrow t \rceil = t$$

$$\lfloor s \rightarrow t \rfloor = s$$

$$s\, t = s$$

Scott D-infinity model (1/2)

- Another labeled λ-calculus was considered to study Scott D-infinity model [Hyland-Wadsworth, 74]

- D-infinity projection functions on each subterm (n is any integer):

$$M, N, \ldots ::= x^n \mid (MN)^n \mid (\lambda x.M)^n$$

- Conversion rule is:

$$((\lambda x.M)^{n+1} N)^p \longrightarrow M\{x := N_{[n]}\}_{[n][p]}$$

$n + 1$ is **degree** of redex

$$U_{[m][n]} = U_{[p]} \quad \text{where} \quad p = \min\{m, n\}$$

$$x^n \{x := M\} = M_{[n]}$$

Scott D-infinity model (2/2)

- **Proposition** Hyland-Wadsworth calculus is derivable from labeled calculus by simple homomorphism on labels.

  Proof Assign an integer to any atomic letter and take:

$$\mathsf{h}(\alpha\beta) = \min\{\mathsf{h}(\alpha),\ \mathsf{h}(\beta)\}$$

$$\mathsf{h}(\lceil\alpha\rceil) = \mathsf{h}(\lfloor\alpha\rfloor) = \mathsf{h}(\alpha) - 1$$

- Redex degrees are bounded by maximum of labels in initial term. therefore a finite number of redex families

- **Proposition** Hyland-Wadsworth calculus strongly normalizes.

Conclusion

- **many** proofs of strong normalization for various calculi
- these proofs look often **magic**
- but intuition is

GFD theorem $\equiv$ strong normalization

- more properties on redex families + labeled calculus
  - standardization theorem
  - completeness of inside-out reductions
  - compactness of main theorems about syntax
  - stability of redexes and sequentiality
  - optimal reductions and relation to Girard's GOI