Strongly Connected Components in graphs, formal proof of Tarjan1972 algorithm

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Plan

- motivation
- algorithm
- pre-/post-conditions
- imperative programs
- conclusion

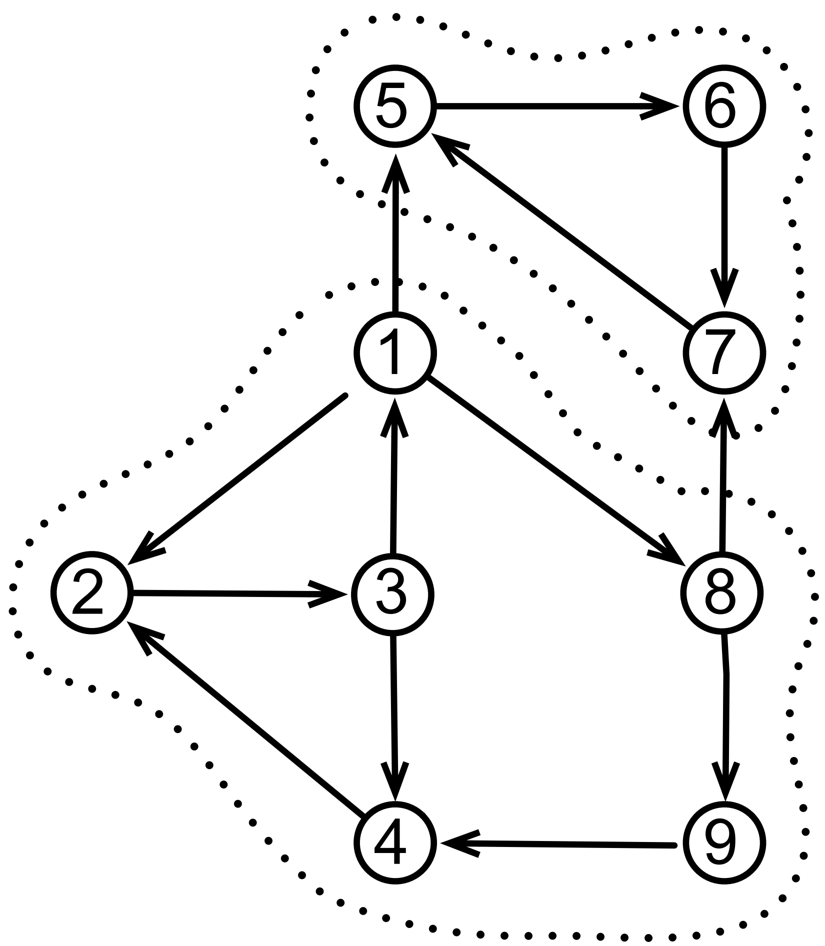
.. joint work (in progress) with Ran Chen

Motivation

- nice algorithms should have simple formal proofs
- to be fully published in articles or journals
- how to publish formal proofs ?
- Coq proofs seem to me unreadable (by normal human)
- Why3 allows mix of automatic and interactive proofs
- first-order logic is easy to understand
- algorithms on graphs = a good testbed

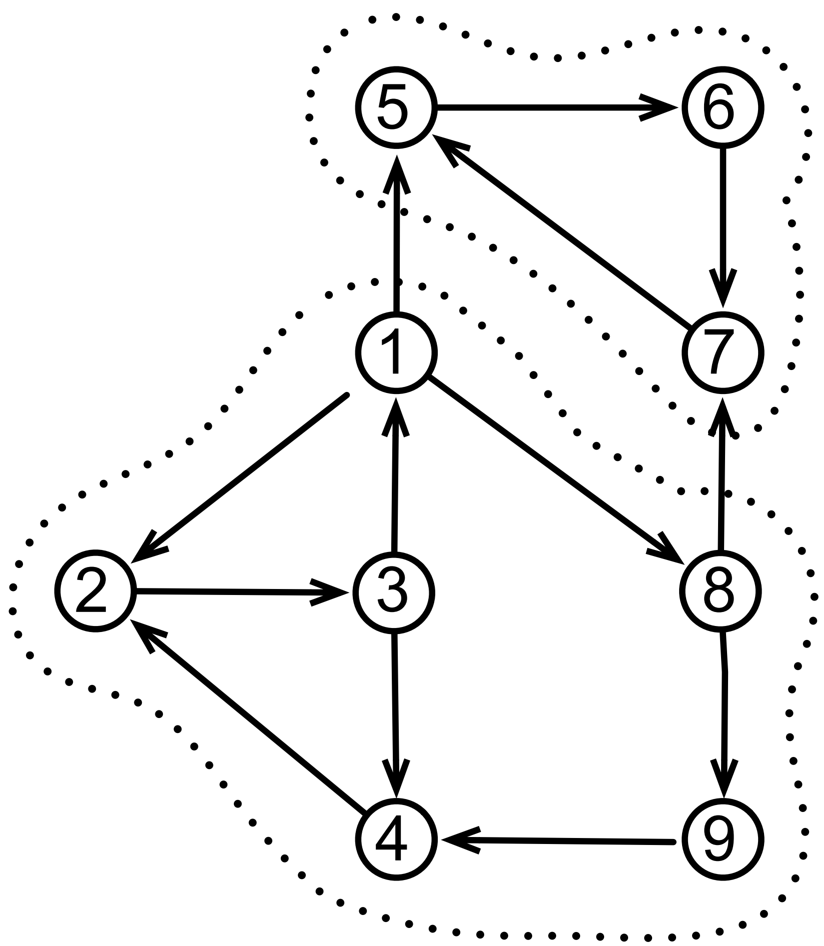
A one-pass linear-time algorithm

The algorithm (1/3)



- depth-first search algorithm
- with pushing non visited vertices into a working stack
- and computing oldest vertex reachable by at most a single « back-edge »
- when that oldest vertex is equal to currently visited vertex, a new strongly connected component is in the working stack on top of current vertex.
- then pop working stack until currently visited vertex

The algorithm (2/3)

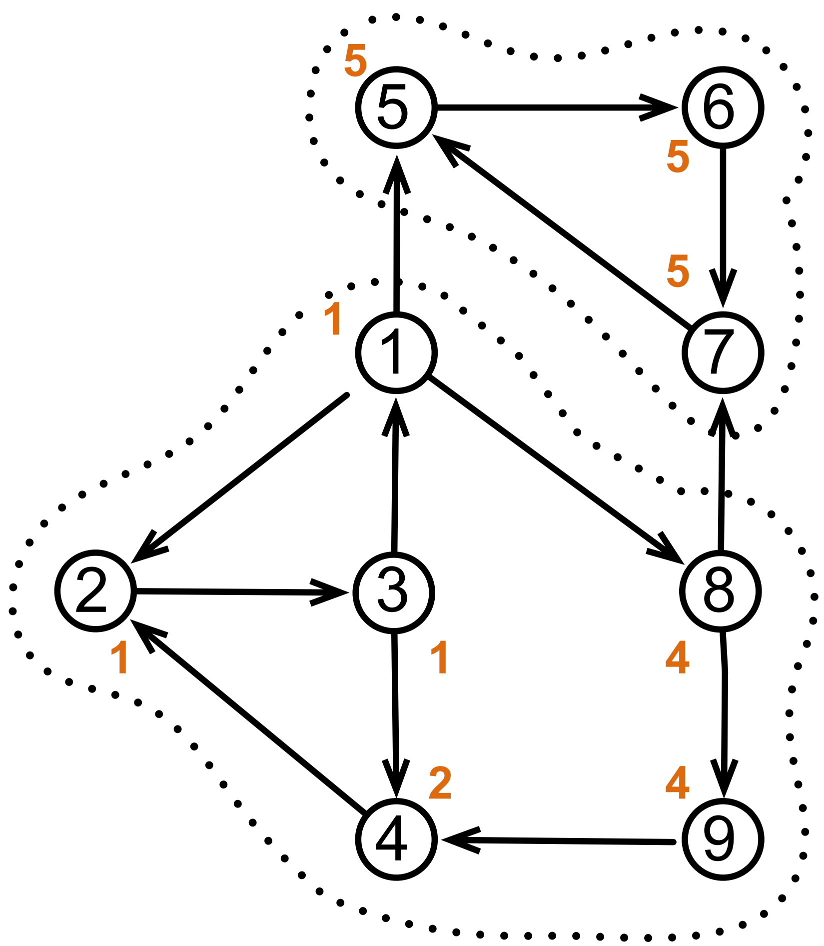


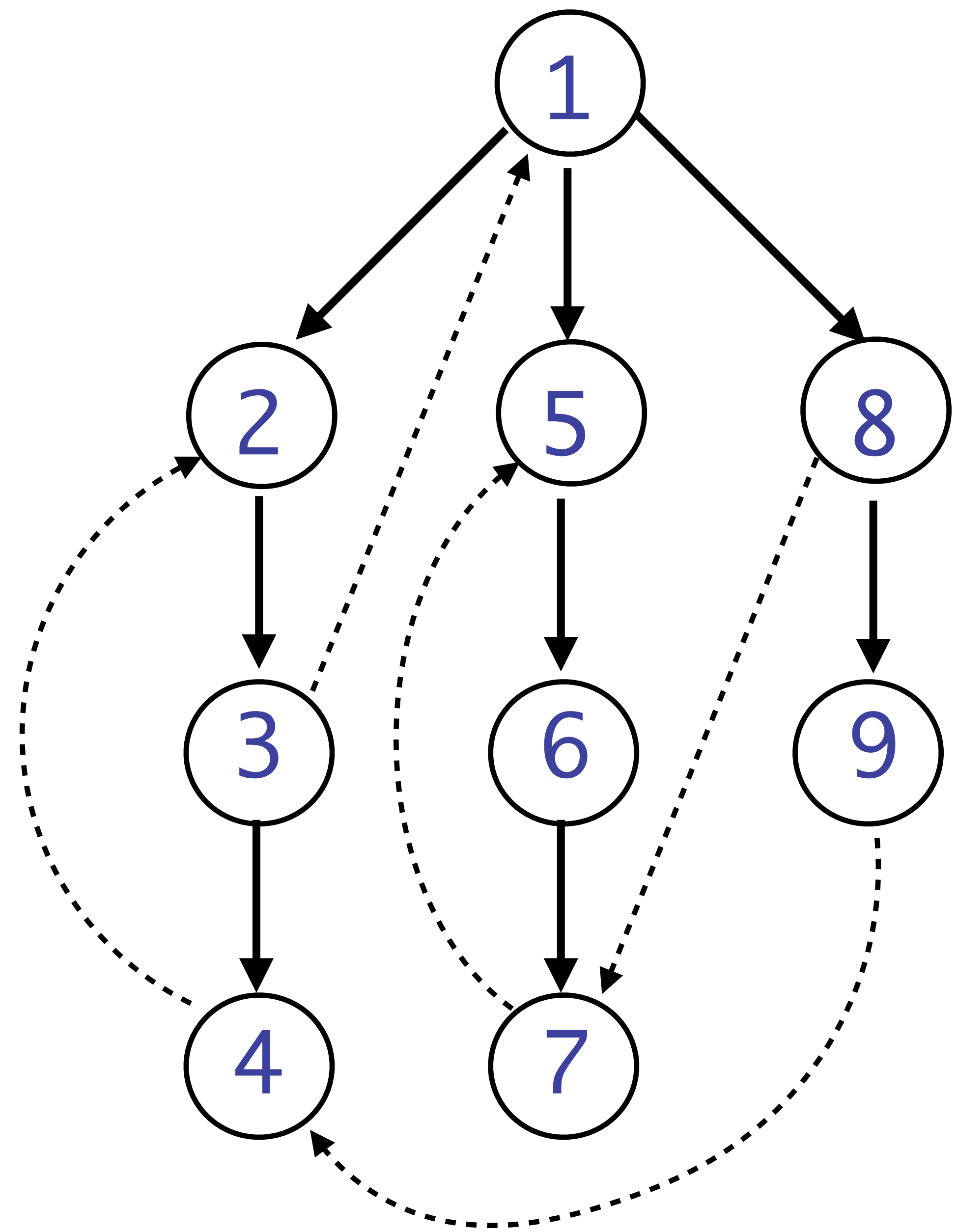
les valeurs successives de la pile

1	1	1	1	1	1	1	1	1	0
	2	2	2	2	2	2	2	2	1
		3	3	3	3	3	3	3	2
			4	4	4	4	4	4	3
				5	5	5	8	8	4
					6	6		9	5
						7			6

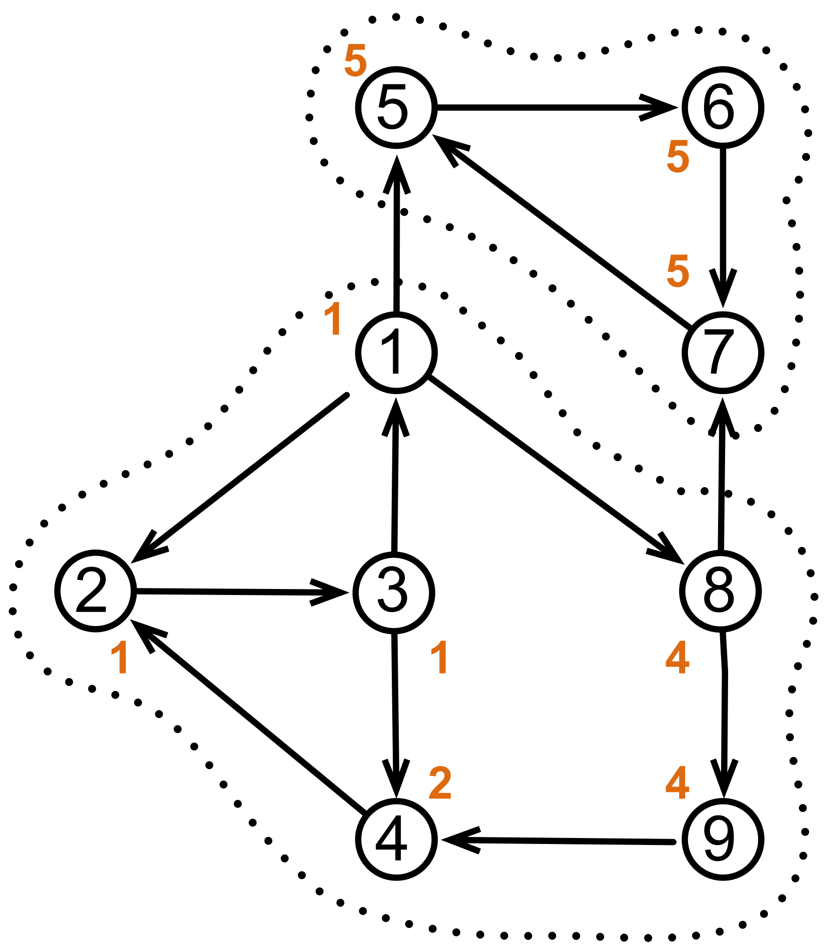
sens croissant du rang

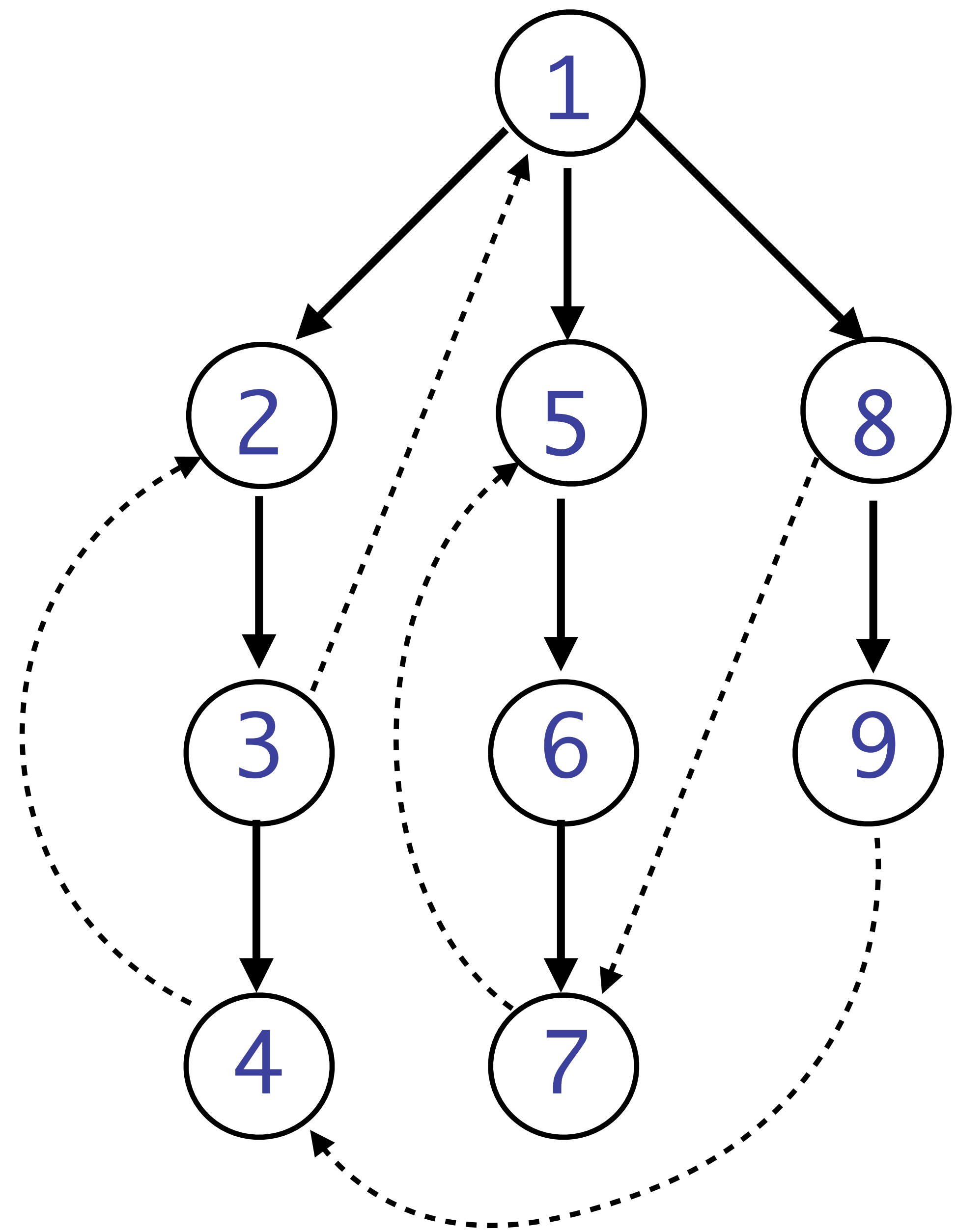
The algorithm (2/3)



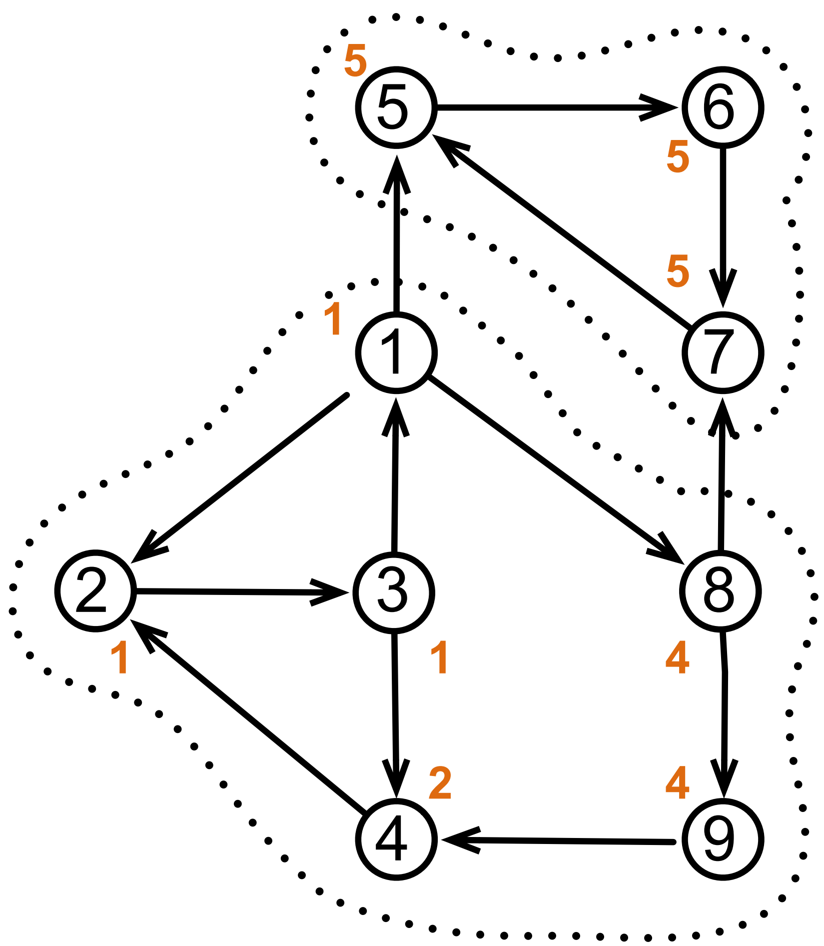


The algorithm (2/3)





The algorithm (2/3)



les valeurs successives de la pile

									rang
1	1	1	1	1	1	1	1	1	0
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		3	3	3	3	3	3	3	2
			4	4	4	4	4	4	3
				5	5	5	8	8	4
					6	6		9	5
						7			6

sens croissant du rang

The algorithm (3/3)

```
let rec printSCC (x: int) (s: Stack.t)
    (num: array int) (serialnb: ref int) =
  Stack.push x s;
  serialnb := !serialnb + 1;
  num[x] ← !serialnb;
  let min = ref num[x] in
  foreach y in (successors x) do
    let m = if num[y] = 0
      then printSCC y s num serialnb
      else num[y] in
    min := Math.min m !min
  done;
```

```
if !min = num[x] then begin
  repeat
    let y = Stack.pop s in
    Printf.printf "%d " y;
    num[y] ← max_int;
    if y = x then break;
  done;
  Printf.printf "\n";
  min := max_int;
end;
return !min;
```

- print each component on a line

Proof in algorithms book (1/2)

- consider the spanning trees (forest)
- tree structure of strongly connected components
- 2-3 lemmas about ancestors in spanning trees

LEMMA 10. *Let v and w be vertices in G which lie in the same strongly connected component. Let F be a spanning forest of G generated by repeated depth-first search. Then v and w have a common ancestor in F. Further, if u is the highest numbered common ancestor of v and w, then u lies in the same strongly connected component as v and w.*

$$LOWLINK(x) = \min\left(\{num[x]\} \;\cup\; \{num[y] \mid x \stackrel{*}{\Longrightarrow}\hookrightarrow y \wedge x \text{ and } y \text{ are in same connected component}\} \right)$$

LEMMA 12. *Let G be a directed graph with* LOWLINK *defined as above relative to some spanning forest F of G generated by depth-first search. Then v is the root of some strongly connected component of G if and only if* LOWLINK $(v) = v$.

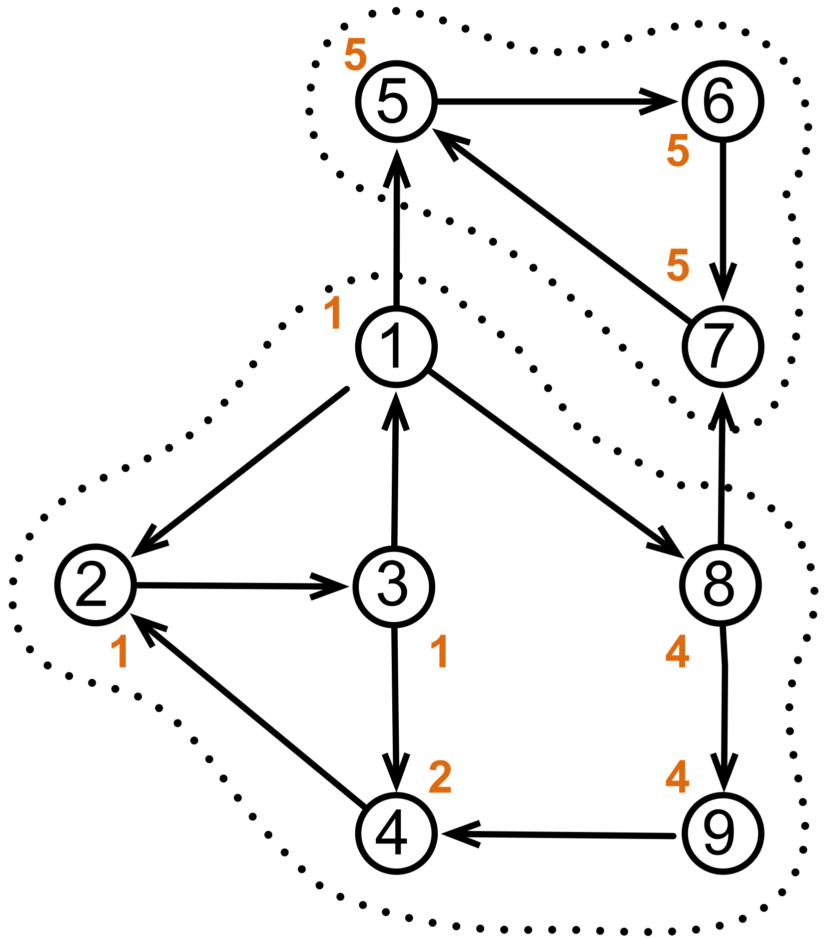
Proof in algorithms book (2/2)

- give the program

- proof ↔ program

- that part of the proof is very informal

The algorithm (bis)



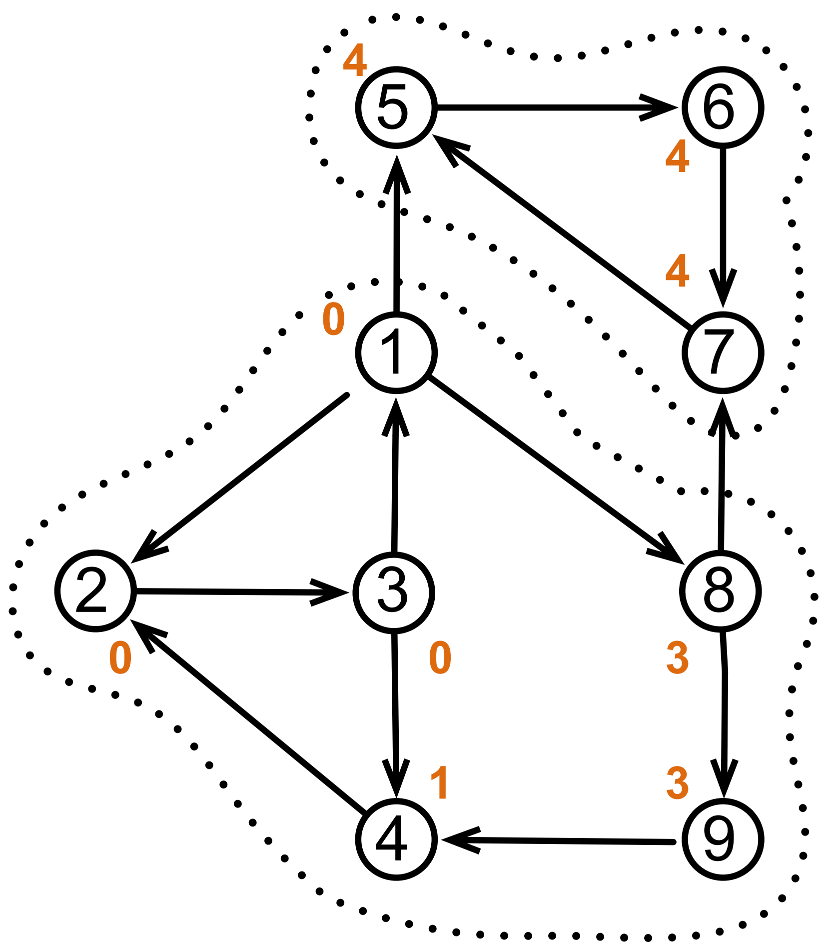
les valeurs successives de la pile

1	1	1	1	1	1	1	1	1		0
	2	2	2	2	2	2	2	2		1
		3	3	3	3	3	3	3		2
			4	4	4	4	4	4		3
				5	5	5	8	8		4
					6	6		9		5
						7				6

sens croissant du rang

$$\mathit{LOWLINK}(x) = \min\ (\ \{\mathit{num}[x]\}\ \cup\ \{\mathit{num}[y]\ \mid\ x \overset{*}{\Longrightarrow}\hookrightarrow y \wedge x \text{ and } y \text{ are in same connected component}\}\ )$$

The algorithm (ter)



les valeurs successives de la pile

									rang
1	1	1	1	1	1	1	1	1	0
	2	2	2	2	2	2	2	2	1
		3	3	3	3	3	3	3	2
			4	4	4	4	4	4	3
				5	5	5	8	8	4
					6	6		9	5
						7			6

sens croissant du rang

$$LOWLINK(x) = \min\ (\ \{rank[y]\ \mid\ x \overset{*}{\Longrightarrow}\hookrightarrow y \wedge x \text{ and } y \text{ are in same connected component}\}\ )$$

Our program (1/3)

- a functional version with lists and finite sets
- the working stack is a list

```
function rank (x: vertex) (s: list vertex): int =
  match s with
  | Nil → max_int()
  | Cons y s' → if x = y && not (lmem x s') then length s' else rank x s'
  end

function max_int (): int = cardinal vertices
```

Our program (1/3)

- a functional version with lists and finite sets
- the working stack is a list

```
let rec split (x : α) (s: list α) : (list α, list α) =
returns{(s1, s2) → s1 ++ s2 = s}
returns{(s1, _) → lmem x s → is_last_of x s1}
  match s with
  | Nil → (Nil, Nil)
  | Cons y s' → if x = y then (Cons x Nil, s') else
        let (s1', s2) = split x s' in
         ((Cons y s1'), s2)
  end
```

Our program (2/3)

- *blacks*, *grays* are sets of vertices; *sccs* is a set of sets of vertices
- naming conventions:

  x, y, z for vertices; *b* for black sets; *s* for stacks;

  cc for connected components;

  sccs for sets of connected components

```
let rec dfs1 x blacks (ghost grays) stack sccs =
  let m = rank x (Cons x stack) in
  let (m1, b1, s1, sccs1) =
    dfs' (successors x) blacks (add x grays) (Cons x stack) sccs in
  if m1 ≥ m then
    let (s2, s3) = split x s1 in
    (max_int(), add x b1, s3, add (elements s2) sccs1)
  else
    (m1, add x b1, s1, sccs1)
```

Our program (3/3)

```
with  dfs' roots blacks (ghost grays) stack sccs =
  if is_empty roots then
    (max_int(), blacks, stack, sccs)
  else
    let x = choose roots in
    let roots' = remove x roots in
    let (m1, b1, s1, sccs1) =
      if lmem x stack then
        (rank x stack, blacks, stack, sccs)
      else if mem x blacks then
        (max_int(), blacks, stack, sccs)
      else
        dfs1 x blacks grays stack sccs in
    let (m2, b2, s2, sccs2) =
      dfs' roots' b1 grays s1 sccs1 in
    (min m1 m2, b2, s2, sccs2)
```

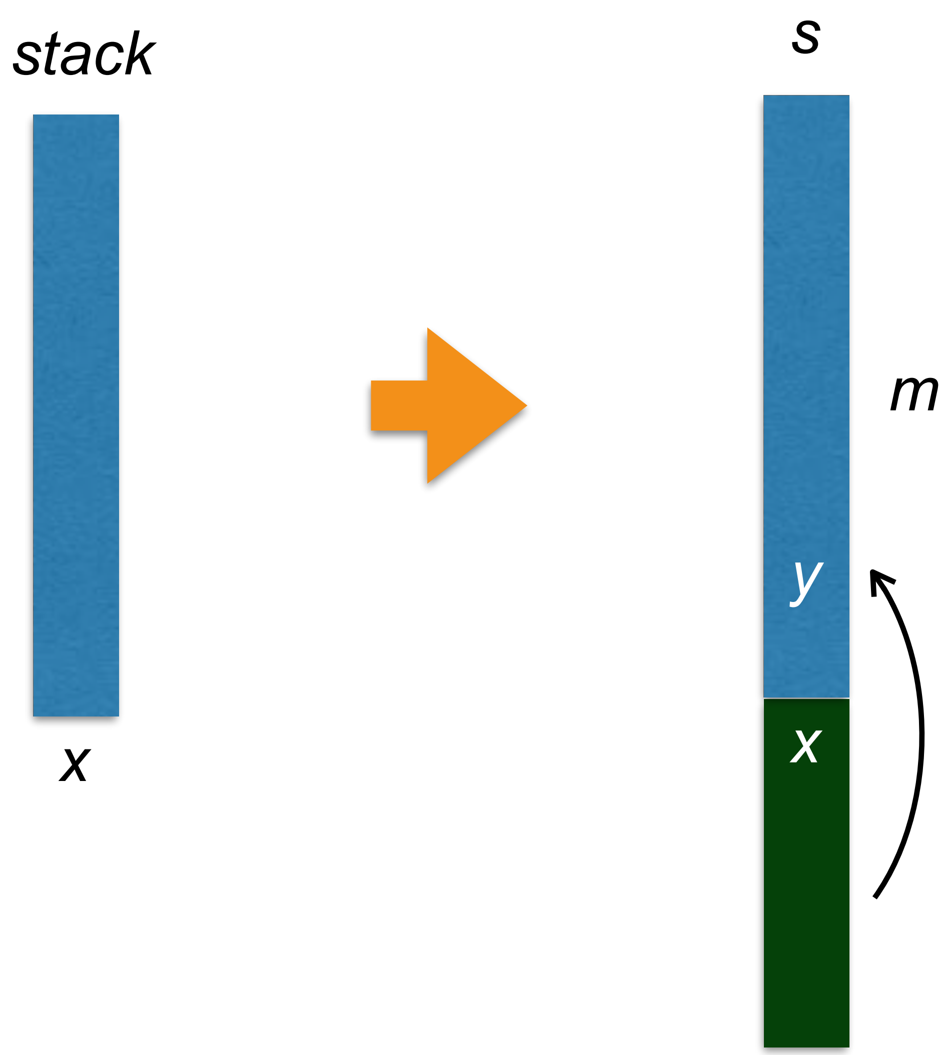
Pre-/Post-conditions

Pre/Post-conditions (1/3)

```
let rec dfs1 x blacks (ghost grays) stack sccs =
requires{mem x vertices} (* R1 *)
requires{access_to grays x} (* R2 *)
requires{not mem x (union blacks grays)} (* R3 *)
```

```
(* monotony *)
returns{(_, b, s, _) → ∃s'. s = s' ++ stack ∧ subset (elements s') b} (* M1 *)
returns{(_, b, _, _) → subset blacks b} (* M2 *)
returns{(_, _, _, sccs_n) → subset sccs sccs_n} (* M3 *)
```

Pre/Post-conditions (2/3)



sccs	$\subseteq$	sccs_n
blacks	$\subseteq$	b

m	$\leq$	rank y stack
m	$\leq$	rank x stack

Pre/Post-conditions (3/3)

```
with  dfs' roots blacks (ghost grays) stack sccs =
requires{subset roots vertices} (* R1 *)
requires{∀x. mem x roots → access_to grays x} (* R2 *)




(* post conditions *)
returns{(_, b, _, _) → subset roots (union b grays)} (* E1 *)
returns{(m, _, s, _) → ∀x. mem x roots → m ≤ rank x s} (* E2 *)
returns{(m, _, s, _) → m = max_int() ∨ ∃x. mem x roots ∧ rank_of_reachable m x s} (
returns{(m, _, s, _) → ∀y. crossedgeto s y stack → m ≤ rank y stack} (* E4 *)
(* monotony *)
returns{(_, b, s, _) → ∃s'. s = s' ++ stack ∧ subset (elements s') b} (* M1 *)
returns{(_, b, _, _) → subset blacks b} (* M2 *)
returns{(_, _, _, sccs_n) → subset sccs sccs_n} (* M3 *)
```

Graphs

```
type vertex
constant vertices: set vertex
function successors vertex : set vertex
axiom successors_vertices:
  ∀x. mem x vertices → subset (successors x) vertices
predicate edge (x y: vertex) =
  mem x vertices ∧ mem y (successors x)
```

Paths

```
inductive path vertex (list vertex) vertex =
  | Path_empty:
      ∀x: vertex. path x Nil x
  | Path_cons:
      ∀x y z: vertex, l: list vertex.
      edge x y → path y l z → path x (Cons x l) z
predicate reachable (x z: vertex) =
  ∃l. path x l z
```

```
predicate in_same_scc (x z: vertex)  =
  reachable x z ∧ reachable z x
predicate is_subscc (s: set vertex) =
  ∀x z. mem x s → mem z s → in_same_scc x z
predicate is_scc (s: set vertex) =
  is_subscc s ∧ (∀s'. subset s s' → is_subscc s' → s == s')
```

Invariants (1/4)

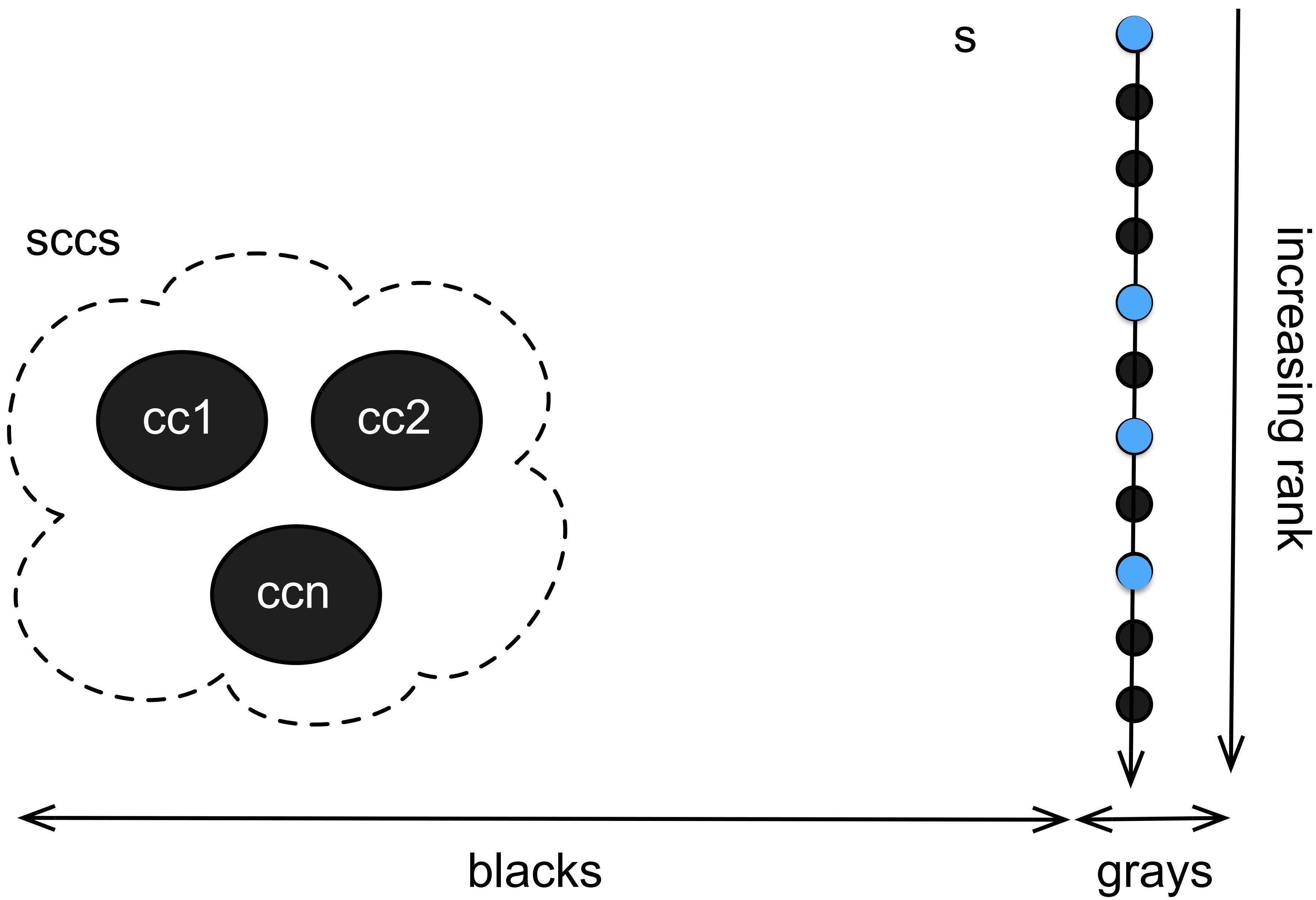
```
predicate no_black_to_white (blacks grays: set vertex) =
  ∀x x'. edge x x' → mem x blacks → mem x' (union blacks grays)

predicate wff_color (blacks grays: set vertex) (s: list vertex)
  (sccs: set (set vertex)) =
  inter blacks grays = empty ∧
  (elements s) == union grays (diff blacks (set_of sccs)) ∧
  (subset (set_of sccs) blacks) ∧
  no_black_to_white blacks grays
```

$$blacks \bigcap grays = \emptyset$$

$$elements\ s = grays \bigcup blacks - (set\_of\ \ sccs)$$

$$(set\_of\ \ sccs) \subseteq blacks$$

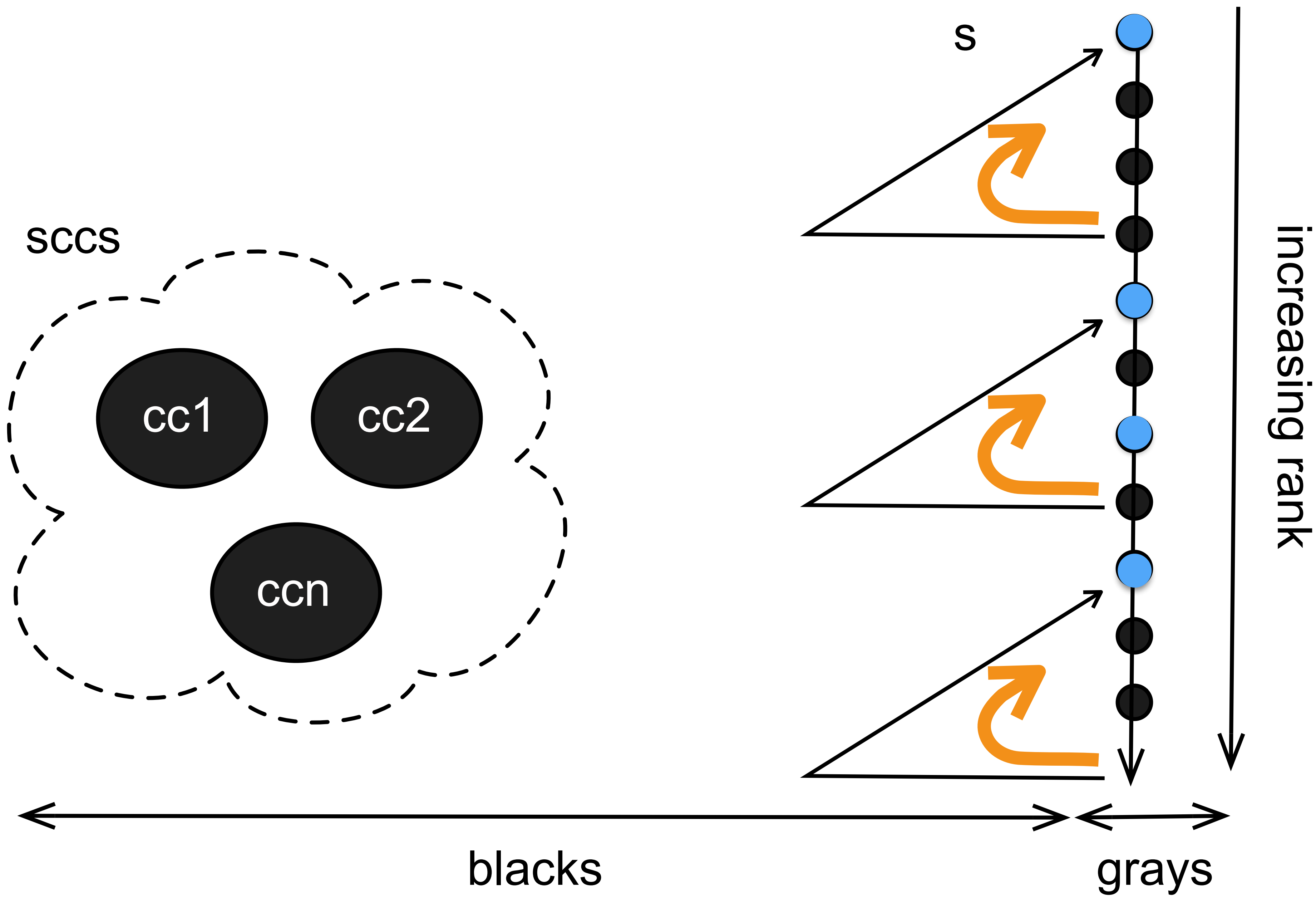
Invariants (2/4)



Invariants (3/4)

```
predicate wff_stack (blacks grays: set vertex) (s: list vertex)
  (sccs: set (set vertex)) =

  wff_color blacks grays s sccs ∧
  simplelist s ∧
  subset (elements s) vertices ∧

  (∀x y. mem x grays → lmem y s →
      rank x s ≤ rank y s → reachable x y) ∧

  (∀y. lmem y s → ∃x. mem x grays ∧
      rank x s ≤ rank y s ∧ reachable y x)
```

Invariants (4/4)



Assertions

```
let m = rank x (Cons x stack) in
let (m1, b1, s1, sccs1) =
dfs' (successors x) blacks (add x grays) (Cons x stack) sccs in

if m1 ≥ m then begin
  let (s2, s3) = split x s1 in
  assert{s3 = stack};
  assert{subset (elements s2) (add x b1)};
  assert{is_subscc (elements s2) ∧ mem x (elements s2)};
  assert{∀y. in_same_scc y x → mem y (elements s2)};
  assert{is_scc (elements s2)};

  (max_int(), add x b1, s3, add (elements s2) sccs1) end
else  begin


  (m1, add x b1, s1, sccs1) end
```

Assertions

```
assert {∀y. in_same_scc y x → mem y (elements s2)};
```

- Coq proof: there exists x', y' with $x' \in s2 \; \wedge \; y' \notin s2 \; \wedge$ edge $x' \; y'$ and x', y' are in same strongly connected component as x

$y' \in s3 =$ stack

$x' = x$ impossible because $m1 \leq$ rank y' s1 $<$ rank x s1

$x' \neq x$ impossible because crossedge

$y' \in$ sccs impossible because sccs disjoint from stack

y' is white

$x' = x$ impossible because successors are black

$x' \neq x$ impossible because no black to white

Pre/Post-conditions (1/3)

```
let rec dfs1 x blacks (ghost grays) stack sccs =
requires{mem x vertices} (* R1 *)
requires{access_to grays x} (* R2 *)
requires{not mem x (union blacks grays)} (* R3 *)
(* invariants *)
requires{wff_stack blacks grays stack sccs} (* I1a *)
requires{∀cc. mem cc sccs ↔ subset cc blacks ∧ is_scc cc} (* I2a *)
returns{(_, b, s, sccs_n) → wff_stack b grays s sccs_n} (* I1b *)
returns{(_, b, _, sccs_n) → ∀cc. mem cc sccs_n ↔ subset cc b ∧ is_scc cc} (* I2b *)
(* post conditions *)
returns{(_, b, _, _) → mem x b} (* E1 *)
returns{(m, _, s, _) → m ≤ rank x s} (* E2 *)
returns{(m, _, s, _) → m = max_int() ∨ rank_of_reachable m x s} (* E3 *)
returns{(m, _, s, _) → ∀y. crossedgeto s y stack → m ≤ rank y stack} (* E4 *)
(* monotony *)
returns{(_, b, s, _) → ∃s'. s = s' ++ stack ∧ subset (elements s') b} (* M1 *)
returns{(_, b, _, _) → subset blacks b} (* M2 *)
returns{(_, _, _, sccs_n) → subset sccs sccs_n} (* M3 *)
```

Full proof

- full proof is at http://jeanjacqueslevy.net/why3
- see the file why3session.html
- proof: *185 lines (38 lemmas) including the program texts.*
- *82* proof obligations

  all proved automatically by *Alt-Ergo (1.30), CVC3 (2.4.1), CVC4 (1.4),*

  Eprover (1.9), Spass (3.5),Yices (1.0.4)

  except *5* of manually checked by *Coq (8.6)*

  Coq proofs are *240* lines *(25+20+119+32+44)*

Towards imperative program

Assertions

```
let rec dfs1 x blacks (ghost grays) stack sccs sn num =
requires{sn = cardinal (union grays blacks) ∧ subset (union grays blacks) vertices}
(* invariants *)
requires{wff_num sn num stack} (* I3a *)
returns{(_, _, _, s, _, sn_n, num_n) → wff_num sn_n num_n s} (* I3b *)
(* post conditions *)
returns{(sn_n, m, _, s, _, _, num_n) → sn_n = m = max_int() ∨
                    ∃y. lmem y s ∧ sn_n = num_n[y] ∧ m = rank y s} (* E5 *)

  let m = rank x (Cons x stack) in
  let (n1, m1, b1, s1, sccs1, sn1, num1) =
    dfs' (successors x) blacks (add x grays) (Cons x stack) sccs (sn + 1) num[x ← sn] in
  if n1 ≥ sn then begin
    let (s2, s3) = split x s1 in
    (max_int(), max_int(), add x b1, s3, add (elements s2) sccs1, sn1, num1) end
  else
    (n1, m1, add x b1, s1, sccs1, sn1, num1)
```

Assertions

```
predicate wff_num (sn: int) (num: map vertex int) (s: list vertex) =
  (∀x. num[x] < sn ≤ max_int())  ∧
  (∀x y. lmem x s → lmem y s → num[x] ≤ num[y] ↔ rank x s ≤ rank y s)
```

Assertions

```
let rec dfs1 x blacks (ghost grays) stack sccs sn num =
  let m = rank x (Cons x stack) in
  let n = !sn in
  incr sn; num := !num[x ← n];
  let (n1, m1, b1, s1, sccs1) =
    dfs' (successors x) blacks (add x grays) (Cons x stack) sccs sn num in
  assert{n1 ≥ n ↔ m1 ≥ m}; (* *)
  if n1 ≥ n then begin
    let (s2, s3) = split x s1 num in
    assert{s3 = stack};
    assert{subset (elements s2) (add x b1)};
    assert{is_subscc (elements s2) ∧ mem x (elements s2)};
    assert{∀y. in_same_scc y x → mem y (elements s2)};
    assert{is_scc (elements s2)};
    (max_int(), max_int(), add x b1, s3, add (elements s2) sccs1) end
  else begin
    assert{∃y. mem y grays ∧ rank y s1 < rank x s1 ∧ reachable x y};
   (n1, m1, add x b1, s1, sccs1) end
```

Missing

- implementation of graphs
- vertices as integers in an array
- successors as lists for every vertex

- see `http://jeanjacqueslevy.net/why3`

Conclusion

Conclusion

- readable proofs ?

- simple algorithms should have simple proofs

→ to be shown with a good formal precision

- compare with other proof systems (without automatic provers?)

- further algorithms (in next talks ?)
  - graphs represented with arrays + lists
  - topological sort, articulation points, sccK, sscT

- Why3 is a beautiful system but not so easy to use !