Can we make readable formal proofs ?

Semi-automatic proofs about graph algorithms

Jean-Jacques Lévy

Inria

Beijing Iscas, 2017-12-08,
Shanghai ECNU, 2017-12-11,
Suzhou USTC, 2017-12-13
Tianjin Nankai, 2017-12-15

Plan

- motivation
- algorithm
- formal proof
- other systems
- conclusion

.. joint work (in progress) with **Ran Chen** [VSTTE 2017])

also cooperation with Cyril Cohen, Laurent Théry, Stephan Merz

DFS in Sedgewick in C

```
int val[maxV]; int id = 0;
visit(int k)
  {
    struct node *t;
    val[k] = ++id;
    for (t = adj[k]; t != z; t = t->next)
      if (val[t->v] == 0) visit(t->v);
  }
listdfs()
  {
    int k;
    for (k = 1; k <= V; k++) val[k] = 0;
    for (k = 1; k <= V; k++)
      if (val[k] == 0) visit(k);
  }
```

Imperative style

DFS in Cormen in pseudocode

```
DFS(G)
for each vertex u ∈ V[G]
     do color[u] ← WHITE
        π[u] ← NIL
time ← 0
for each vertex u ∈ V[G]
     do if color[u] = WHITE
           then DFS-VISIT(u)

DFS-VISIT(u)
color[u] ← GRAY                  ▷ White vertex u has just been discovered.
d[u] ← time ← time + 1
for each v ∈ Adj[u]              ▷ Explore edge (u, v).
     do if color[v] = WHITE
           then π[v] ← u
                DFS-VISIT(v)
color[u] ← BLACK                 ▷ Blacken u; it is finished.
f[u] ← time ← time + 1
```

Imperative style

DFS in Why3 language

```
type vertex
constant vertices: set vertex
function successors vertex : set vertex
axiom successors_vertices:
    forall x. mem x vertices -> subset (successors x) vertices
predicate edge (x y: vertex) = mem x vertices /\ mem y (successors x)
```

DFS in Why3 language

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```

- a **functional** version with **finite sets**

```
let rec dfs (roots visited: set vertex): set vertex =
 if is_empty roots then visited else
  let x = choose roots in
  let roots' = remove x roots in
  if mem x visited then
    dfs roots' visited
  else
    let v’ = dfs (successors x) (add x visited) in
    dfs roots’ (union visited v’)

let dfs_main (roots: set vertex) : set vertex =
  dfs roots empty
```

Functional programming

DFS in Why3 language

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```

- a **functional** version with **finite sets** and type inference

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 if is_empty roots then visited else
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  dfs roots empty
```

Functional programming

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- a **functional** version with **finite sets** and type inference

```
let rec dfs r v =
 if is_empty r then v else
  let x = choose r in
  let r' = remove x r in
  if mem x v then
    dfs r' v
  else
    let v' = dfs (successors x) (add x v) in
    dfs r' (union v v')

let dfs_main roots =
  dfs roots empty
```

Functional programming

DFS in imperative style

- data are efficiently implemented:
  - **finite sets** → lists or arrays
  - direct access from nodes to their successors → array of lists of successors
- programming languages problems:
  - array bounds checking
  - mutable variables: uneasy notations, frame problem
- not treated here

DFS with formal proofs in literature

- for us, formal proofs ➡ checked by computer
- in Isabelle / HOL
- in Coq / ssreflect in the **mathematical components** library

http://math-comp.github.io/math-comp/htmldoc/libgraph.html

https://github.com/math-comp/math-comp/blob/master/mathcomp/ssreflect/fingraph.v

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- proofs to be read by computer
- read these proofs → good training

Formal proofs read by computer

- proofs with many cases
- very long proofs
- getting harder with structured proofs

- automatic provers could simplify
- but often loosing proof certificates

- basic algorithms ➡ readable proofs (by humans)
    - teaching formal methods
    - stressing on methodology for simplification

Random search in Why3 language

```
type vertex
constant vertices: set vertex
function successors vertex : set vertex
axiom successors_vertices:
    forall x. mem x vertices -> subset (successors x) vertices
predicate edge (x y: vertex) = mem x vertices /\ mem y (successors x)
```

- one step of any traversal strategy [dowek, munoz]

```
let rec random_search r v =
   if is_empty r then v else
    let x = choose r in
    let r' = remove x r in
     if mem x v then
       random_search r' v
     else
       random_search (union r' (successors x)) (add x v)

let random_search_main roots =
   random_search roots empty
```

Random search completeness

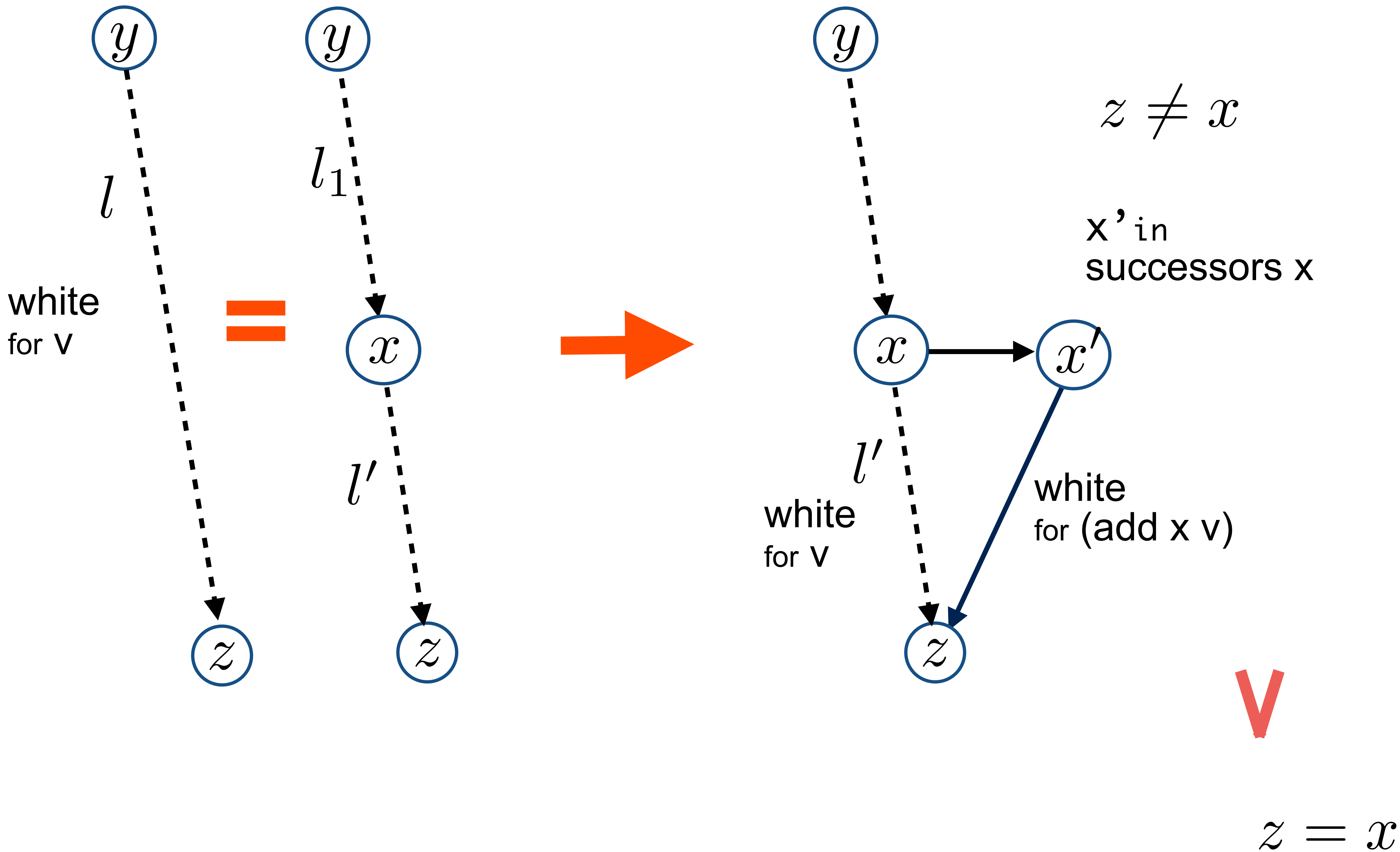
$$\texttt{predicate}\ \ white\_vertex\,(x:\ vertex)\,(v:\ set\ vertex) =\ \neg\,(mem\ x\ v)$$

$$\texttt{predicate}\ \ whitepath\,(x:\ vertex)\,(l : list\ vertex)\,(z : vertex)\,(v : set\ vertex) = \\ path\ x\ l\ z\ \wedge\ (\forall y.\, L.mem\ y\ l\ \rightarrow\ white\_vertex\ y\ v)\ \wedge\ white\_vertex\ z\ v$$

```
let rec random_search r v =
  requires {subset r vertices}
  requires {subset v vertices}
  ensures {subset v result}
  ensures {forall z y l. mem y r -> whitepath y l z v -> mem z (diff result v)}
  if is_empty r then v else
    let x = choose r in
    let r' = remove x r in
     if mem x v then
       random_search r' v
     else
       let v' = random_search (union r' (successors x)) (add x v) in
       assert {forall z y l. mem y r -> whitepath y l z v -> z <> x ->
         whitepath y l z (add x v) \/
         exists x' l'. mem x' (successors x) /\ whitepath x' l' z (add x v)} ;
       v'

let random_search_main roots =
   requires {subset roots vertices}
   assert {forall z y l. mem y r -> path y l z -> mem z result}
   random_search roots empty
```

Random search completeness



Random search soundness

predicate $\mathit{white\_vertex}\,(x:\ \mathit{vertex})\,(v:\ \mathit{set}\ \mathit{vertex}) = \ \neg\,(\mathit{mem}\ x\ v)$

predicate $\mathit{whitepath}\,(x:\ \mathit{vertex})\,(l:\mathit{list}\ \mathit{vertex})\,(z:\mathit{vertex})\,(v:\mathit{set}\ \mathit{vertex}) =$
$\quad \mathit{path}\ x\ l\ z\ \wedge\ (\forall y.\, L.\mathit{mem}\ y\ l\ \rightarrow\ \mathit{white\_vertex}\ y\ v)\ \wedge\ \mathit{white\_vertex}\ z\ v$

```
let rec random_search r v =
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       assert {forall z. mem z (diff v' v) -> z <> x ->
           (exists y l. mem y r'/\ (whitepath y l z v)
        \/ (exists l. whitepath x l z v)}
       v'

let random_search_main roots =
   requires {subset roots vertices}
   assert {forall z. mem z result -> exists y l. mem y r -> path y l z}
   random_search roots empty
```

Random search soundness

predicate $white\_vertex\ (x:\ vertex)\ (v:\ set\ vertex) = \neg\ (mem\ x\ v)$

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$path\ x\ l\ z\ \wedge\ (\forall y.\ L.mem\ y\ l\ \rightarrow\ white\_vertex\ y\ v)\ \wedge\ white\_vertex\ z\ v$

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       assert {forall z. mem z (diff v' v) -> z <> x ->
           (exists y l. (mem y r'/\ (whitepath y l z v
            by whitepath y l z (add x v) ))
        \/ ((exists l. whitepath x l z v)
            by exists y' l'. mem y' (successors x) /\ whitepath y' l' z v))};
       v'

let random_search_main roots =
   requires {subset roots vertices}
   assert {forall z. mem z result -> exists y l. mem y r -> path y l z}
   random_search roots empty
```

Random search soundness

predicate $\mathit{white\_vertex}\ (x : \mathit{vertex})\ (v : \mathit{set\ vertex}) = \neg\ (\mathit{mem}\ x\ v)$

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$\quad \mathit{path}\ x\ l\ z\ \wedge\ (\forall y.\ L.\mathit{mem}\ y\ l\ \rightarrow\ \mathit{white\_vertex}\ y\ v)\ \wedge\ \mathit{white\_vertex}\ z\ v$

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  requires {subset r vertices}
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       v'

let random_search_main roots =
   requires {subset roots vertices}
   assert {forall z. mem z result -> exists y l. mem y r -> path y l z}
   random_search roots empty
```

demo

Other algorithms on graphs

- dfs (recursive), dfs (iterative), bfs
- dfs (imperative)
- dag test
- dfs (recursive with non-black-to-white predicate)
- dfs (undirected graphs with non-white-to-black proof)
- minimum spanning tree (50% done)
- strongly connected components (kosaraju)
- strongly connected components (tarjan)

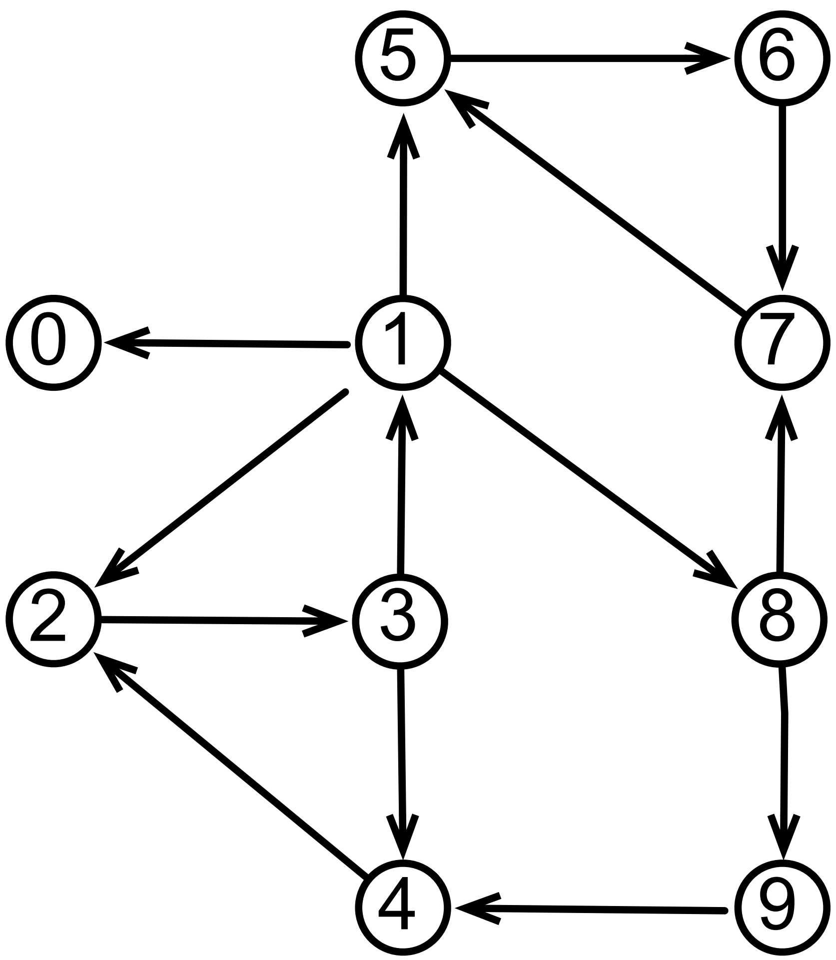
http://jeanjacqueslevy.net/why3

http://pauillac.inria.fr/~levy/why3

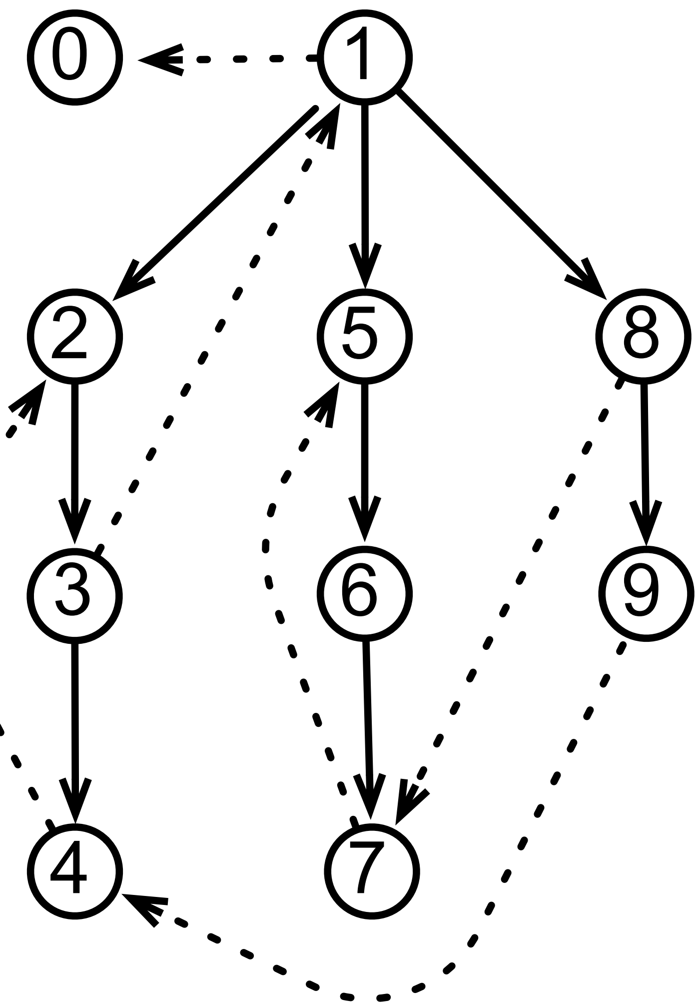
One-pass linear-time algorithm

[tarjan 1972]

Depth-first-search

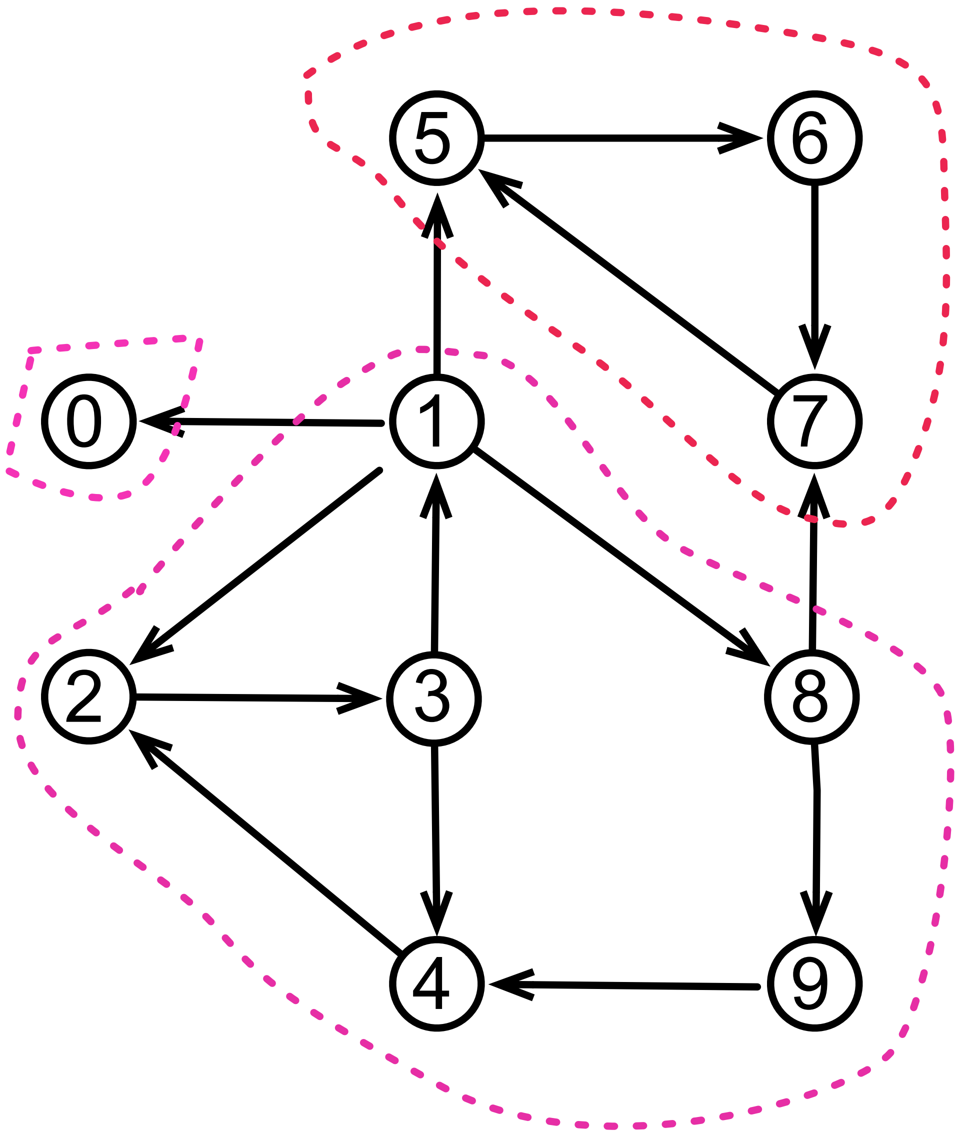


graph

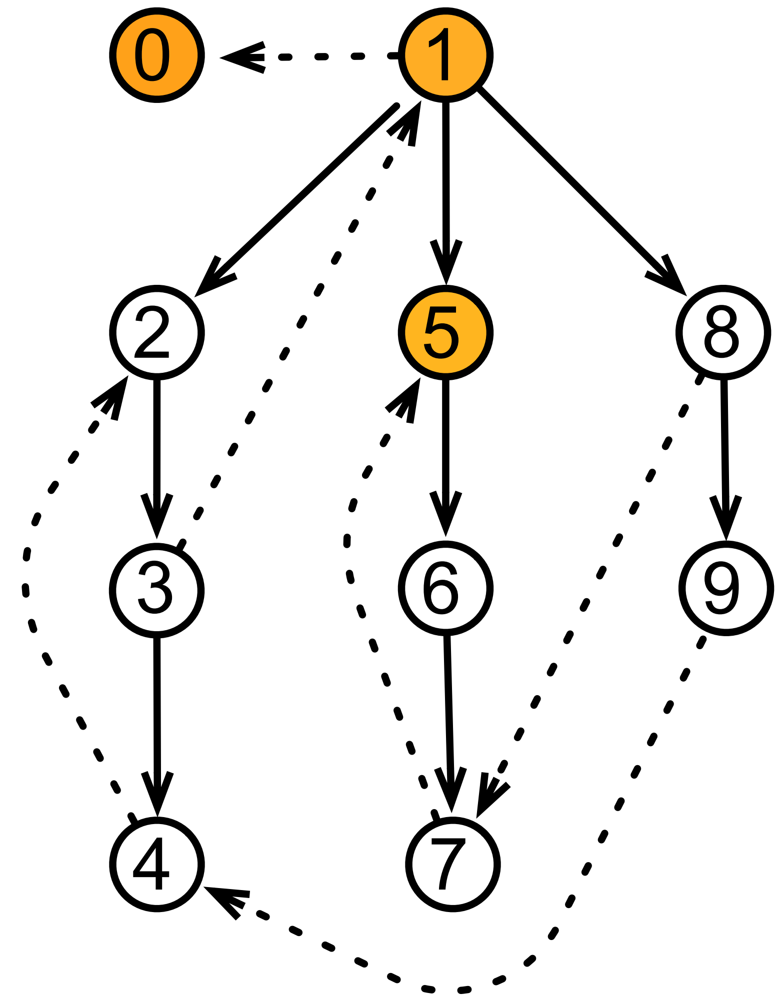


spanning tree (forest)

The algorithm (1/3)

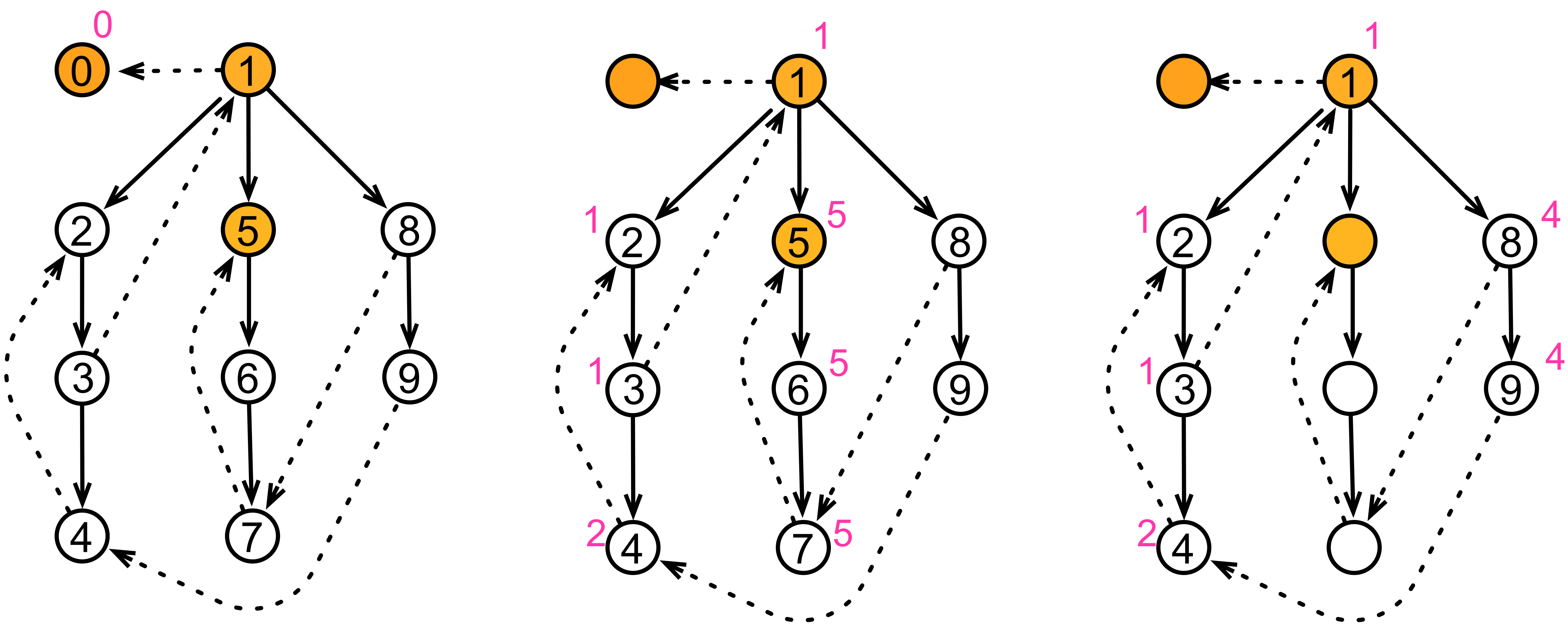


3 SCCs (strongly connected components)



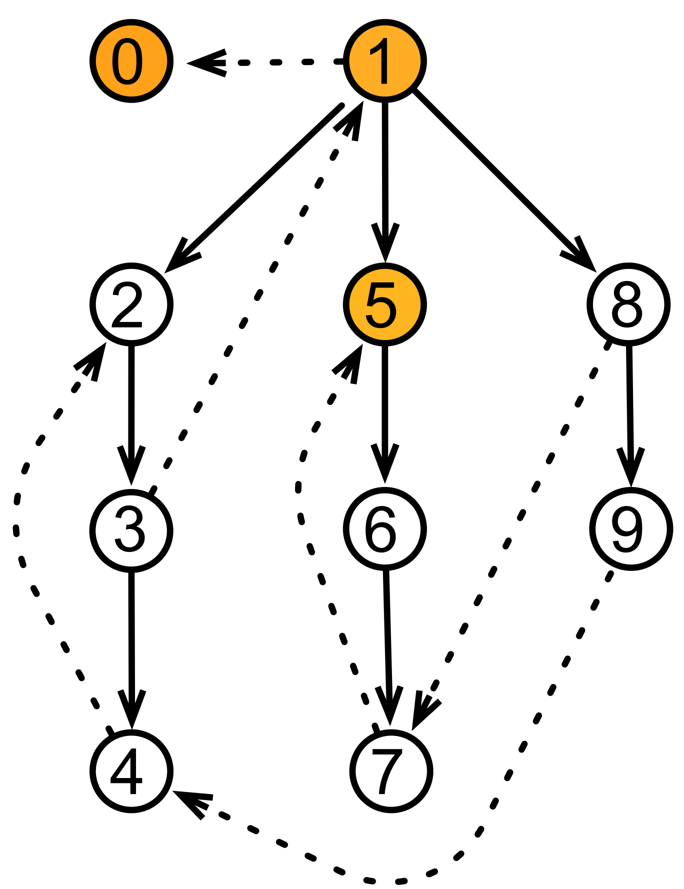
3 vertices are their bases

The algorithm (2/3)



$$LOWLINK(x) = \min \left(\{num[x]\} \;\cup\; \{num[y] \mid x \xrightarrow{*}\dashrightarrow y \;\wedge\; x \text{ and } y \text{ are in same connected component}\} \right)$$

The algorithm (3/3)



successive values of the working stack

increasing rank										
0	0	1	1	1	1	1	1	1	1	1
1			2	2	2	2	2	2	2	2
2				3	3	3	3	3	3	3
3					4	4	4	4	4	4
4						5	5	5	8	8
5							6	6		9
6								7		

The program

```
let rec printSCC (x: int) (s: stack int)
    (num: array int) (sn: ref int) =
  Stack.push x s;
  num[x] ← !sn; sn := !sn + 1;
  let low = ref num[x] in
  foreach y in (successors x) do
    let m = if num[y] = -1
      then printSCC y s num sn
      else num[y] in
    low := Math.min m !low
  done;
```

```
if !low = num[x] then begin
  repeat
    let y = Stack.pop s in
    Printf.printf "%d " y;
    num[y] ← max_int;
    if y = x then break;
  done;
  Printf.printf "\n";
  low := max_int;
end;
return !low;
```

- print each component on a line

Imperative style

Proof in algorithms books (1/2)

- consider the spanning trees (forest)
- tree structure of strongly connected components
- 2-3 lemmas about ancestors in spanning trees

LEMMA 10. *Let v and w be vertices in G which lie in the same strongly connected component. Let F be a spanning forest of G generated by repeated depth-first search. Then v and w have a common ancestor in F. Further, if u is the highest numbered common ancestor of v and w, then u lies in the same strongly connected component as v and w.*

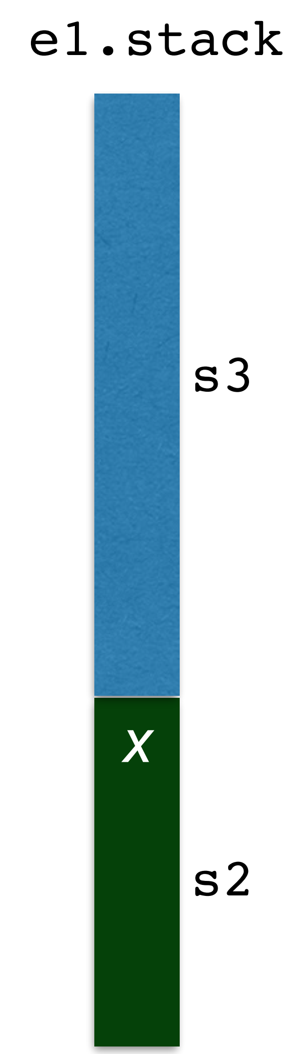
$$\mathit{LOWLINK}(x) = \min\left(\{\mathit{num}[x]\} \;\cup\; \{\mathit{num}[y] \mid x \overset{*}{\Longrightarrow}\hookrightarrow y \wedge x \text{ and } y \text{ are in same connected component}\} \right)$$

LEMMA 12. *Let G be a directed graph with* LOWLINK *defined as above relative to some spanning forest F of G generated by depth-first search. Then v is the root of some strongly connected component of G if and only if* LOWLINK $(v) = v$.

Proof in algorithms book (2/2)

- give the program
- proof ↔ program
- that part of the proof is very informal

Our program (1/3)




```
let rec dfs1 x e =
  let n = e.sn in
  let (n1, e1) = dfs (successors x) (add_stack_incr x e) in
  let (s2, s3) = split x e1.stack in
  if n1 < n then (n1, e1) else
    (max_int(), {stack = s3; sccs = add (elements s2) e1.sccs;
      sn = e1.sn; num = set_max_int s2 e1.num})

with  dfs roots e = if is_empty roots then (max_int(), e) else
  let x = choose roots in
  let roots' = remove x roots in
  let (n1, e1) = if e.num[x] ≠ -1 then (e.num[x], e) else dfs1 x e in
  let (n2, e2) = dfs roots' e1 in (min n1 n2, e2)

let tarjan () =
  let e0 = {stack = Nil; sccs = empty; sn = 0; num = const (-1)} in
  let (_, e') = dfs vertices e0 in e'.sccs
```

returns *LOWLINK*(x) and new environment

Functional programming

Formal proof

using Why3

Plan of proof (1/2)

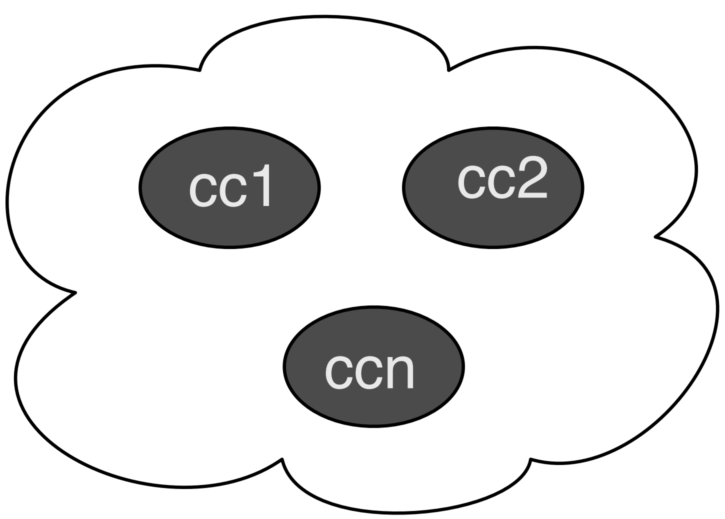
- define **reachability** in graphs and SCCs
- prove a few lemmas about positions in stacks (**ranks**)
- define **invariants** on environments
- give **pre-post conditions** for functions
- add a few intermediate **assertions** in function bodies

- avoid paths, prefer edges

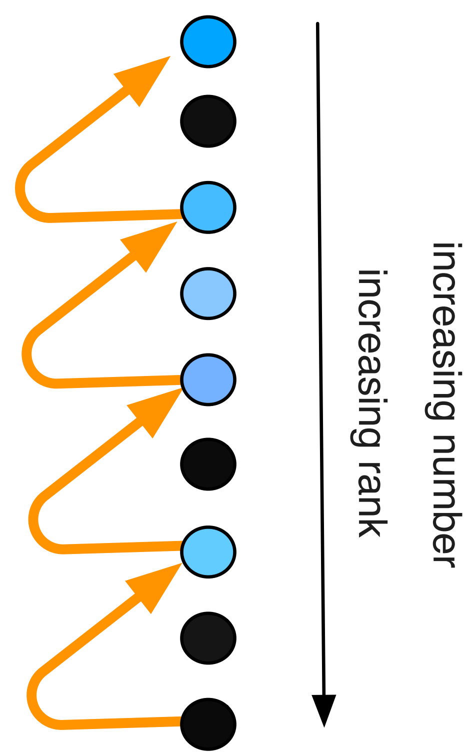
Plan of proof (2/2)

- vertices have colors
  - white = unvisited
  - gray = being visited
  - black = visited
- invariant on environment

sccs



stack



vertex in stack reaches all vertices with higher rank

Invariants

```
type env = {ghost blacks: set vertex; ghost grays: set vertex;
            stack: list vertex; sccs: set (set vertex);
            sn: int; num: map vertex int}
```

Invariants

```
type env = {ghost blacks: set vertex; ghost grays: set vertex;
            stack: list vertex; sccs: set (set vertex);
            sn: int; num: map vertex int}

predicate wf_color (e: env) =
  let {stack = s; blacks = b; grays = g; sccs = ccs} = e in
  subset (union g b) vertices /\
  inter b g == empty /\
  elements s == union g (diff b (set_of ccs)) /\
  subset (set_of ccs) b
```

Invariants

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type env = {ghost blacks: set vertex; ghost grays: set vertex;
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  elements s == union g (diff b (set_of ccs)) /\
  subset (set_of ccs) b

predicate wf_num (e: env) =
  let {stack = s; blacks = b; grays = g; sccs = ccs; sn = n; num = f} = e in
  (forall x. -1 <= f[x] < n <= max_int() \/ f[x] = max_int())  /\
  n = cardinal (union g b) /\
  (forall x. f[x] = max_int() <-> mem x (set_of ccs)) /\
  (forall x. f[x] = -1 <-> not mem x (union g b)) /\
  (forall x y. lmem x s -> lmem y s -> f[x] < f[y] <-> rank x s < rank y s)
```

Invariants

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type env = {ghost blacks: set vertex; ghost grays: set vertex;
            stack: list vertex; sccs: set (set vertex);
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  (forall x y. lmem x s -> lmem y s -> f[x] < f[y] <-> rank x s < rank y s)

predicate no_black_to_white (blacks grays: set vertex) =
  forall x x'. edge x x' -> mem x blacks -> mem x' (union blacks grays)
```

Invariants

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type env = {ghost blacks: set vertex; ghost grays: set vertex;
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  (forall x y. lmem x s -> lmem y s -> f[x] < f[y] <-> rank x s < rank y s)

predicate no_black_to_white (blacks grays: set vertex) =
  forall x x'. edge x x' -> mem x blacks -> mem x' (union blacks grays)

predicate wf_env (e: env) = let {stack = s; blacks = b; grays = g} = e in
  wf_color e /\ wf_num e /\
  no_black_to_white b g /\ simplelist s /\
  (forall x y. mem x g -> lmem y s -> rank x s <= rank y s -> reachable x y) /\
  (forall y. lmem y s -> exists x. mem x g /\ rank x s <= rank y s /\ reachable y x)
```

Pre/Post-conditions

```
let rec dfs1 x e  =
requires {mem x vertices} (* R1 *)
requires {access_to e.grays x} (* R2 *)
requires {not mem x (union e.blacks e.grays)} (* R3 *)
```

:

Pre/Post-conditions

```
let rec dfs1 x e  =
requires {mem x vertices} (* R1 *)
requires {access_to e.grays x} (* R2 *)
requires {not mem x (union e.blacks e.grays)} (* R3 *)
(* invariants *)
requires {wf_env e} (* I1a *)
requires {forall cc. mem cc e.sccs <-> subset cc e.blacks /\ is_scc cc} (* I2a *)
returns {(_, e') -> wf_env e'} (* I1b *)
returns {(_, e') -> forall cc. mem cc e'.sccs <-> subset cc e'.blacks /\ is_scc cc} (* I2b *)
```

Pre/Post-conditions

```
let rec dfs1 x e  =
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requires {access_to e.grays x} (* R2 *)
requires {not mem x (union e.blacks e.grays)} (* R3 *)
(* invariants *)
requires {wf_env e} (* I1a *)
requires {forall cc. mem cc e.sccs <-> subset cc e.blacks /\ is_scc cc} (* I2a *)
returns {(_, e') -> wf_env e'} (* I1b *)
returns {(_, e') -> forall cc. mem cc e'.sccs <-> subset cc e'.blacks /\ is_scc cc} (* I2b *)




(* monotony *)
returns {(_, e') -> subenv e e'}
```

Pre/Post-conditions

```
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(* invariants *)
requires {wf_env e} (* I1a *)
requires {forall cc. mem cc e.sccs <-> subset cc e.blacks /\ is_scc cc} (* I2a *)
returns {(_, e') -> wf_env e'} (* I1b *)
returns {(_, e') -> forall cc. mem cc e'.sccs <-> subset cc e'.blacks /\ is_scc cc} (* I2b *)
```
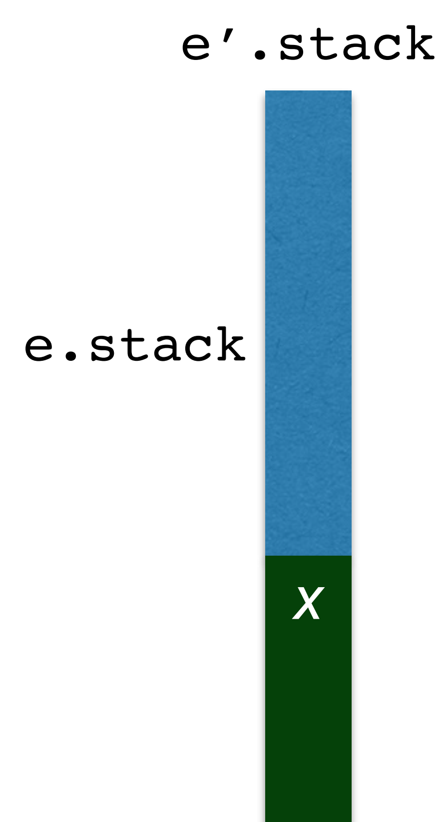

```
(* monotony *)
returns {(_, e') -> subenv e e'}
```

$$e.sccs \subseteq e'.sccs$$

$$e.blacks \subseteq e'.blacks$$

$$e.grays = e'.grays$$



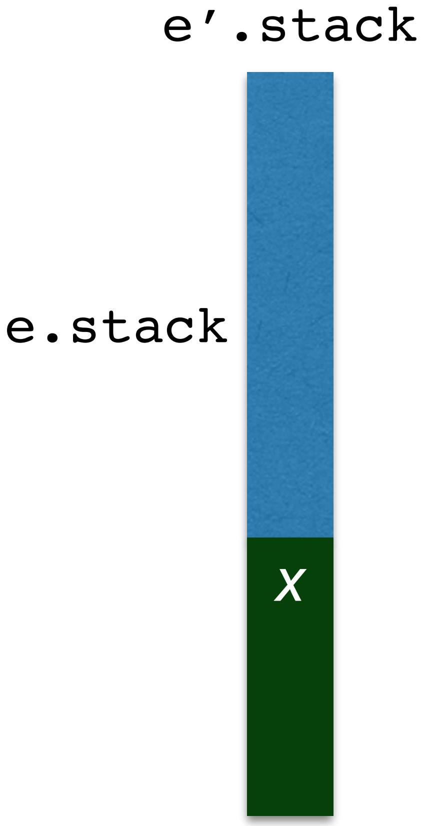
Pre/Post-conditions

```
let rec dfs1 x e  =
requires {mem x vertices} (* R1 *)
requires {access_to e.grays x} (* R2 *)
requires {not mem x (union e.blacks e.grays)} (* R3 *)
(* invariants *)
requires {wf_env e} (* I1a *)
requires {forall cc. mem cc e.sccs <-> subset cc e.blacks /\ is_scc cc} (* I2a *)
returns {(_, e') -> wf_env e'} (* I1b *)
returns {(_, e') -> forall cc. mem cc e'.sccs <-> subset cc e'.blacks /\ is_scc cc} (* I2b *)
(* post-cond *)
returns {(n, e') -> n <= e'.num[x]} (* PC1 *)
returns {(n, e') -> n = max_int() \/ num_of_reachable n x e'} (* PC2 *)
returns {(n, e') -> forall y. xedge_to e'.stack e.stack y -> n <= e'.num[y]} (* PC3 *)
returns {(_, e') -> mem x e'.blacks} (* PC4 *)
(* monotony *)
returns {(_, e') -> subenv e e'}
```

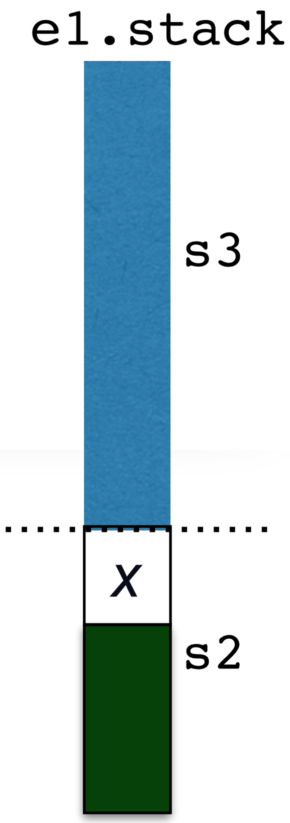
e.sccs $\subseteq$ e'.sccs

e.blacks $\subseteq$ e'.blacks

e.grays = e'.grays



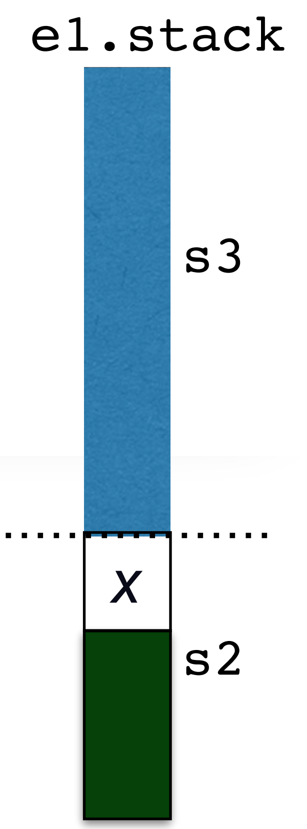
Assertions




```
let n = e.sn in
let (n1, e1) =
    dfs' (successors x) (add_stack_incr x e) in
let (s2, s3) = split x e1.stack in


if n1 < n then begin

  (n1, add_blacks x e1) end
else  begin



  (max_int(), {blacks = add x e1.blacks; grays = e.grays;
     stack = s3; sccs = add (elements s2) e1.sccs;
     sn = e1.sn; num = set_max_int s2 e1.num}) end
```

[http://jeanjacqueslevy.net/why3/graph/abs/scct/1-7/scc.html]

Assertions




```
let n = e.sn in
let (n1, e1) =
    dfs' (successors x) (add_stack_incr x e) in
let (s2, s3) = split x e1.stack in
assert {is_last x s2 /\ s3 = e.stack /\ subset (elements s2) (add x e1.blacks)};
assert {is_subscc (elements s2)};
if n1 < n then begin

  (n1, add_blacks x e1) end
else  begin



  (max_int(), {blacks = add x e1.blacks; grays = e.grays;
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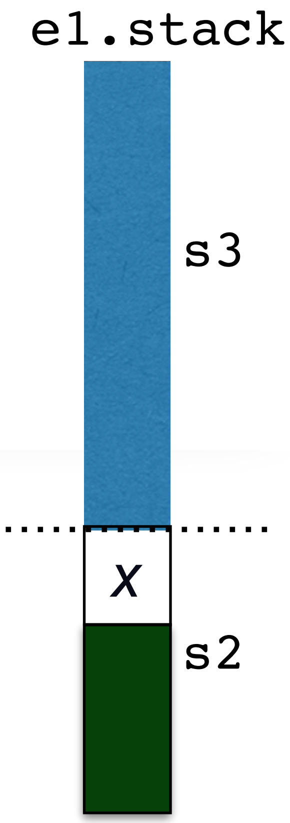


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if n1 < n then begin
  assert {exists y. mem y e.grays /\ lmem y e1.stack /\ e1.num[y] < e1.num[x] /\ reachable x y};
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  (max_int(), {blacks = add x e1.blacks; grays = e.grays;
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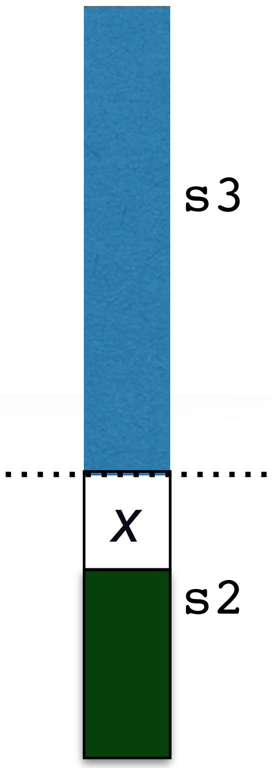
[http://jeanjacqueslevy.net/why3/graph/abs/scct/1-7/scc.html]

Assertions

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else  begin
  assert {forall y. in_same_scc y x -> lmem y s2};
  assert {is_scc (elements s2)};
  assert {inter e.grays (elements s2) = empty by inter e.grays (elements s2) == empty};
  (max_int(), {blacks = add x e1.blacks; grays = e.grays;
     stack = s3; sccs = add (elements s2) e1.sccs;
     sn = e1.sn; num = set_max_int s2 e1.num}) end
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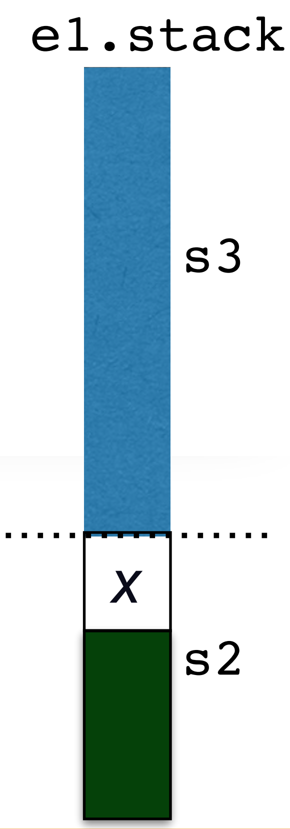
Assertions

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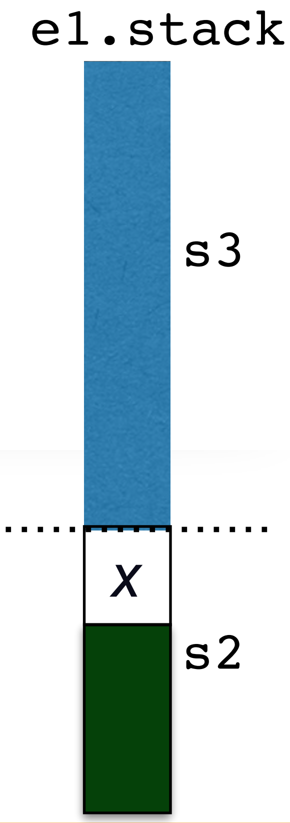
[http://jeanjacqueslevy.net/why3/graph/abs/scct/1-7/scc.html]

Assertions

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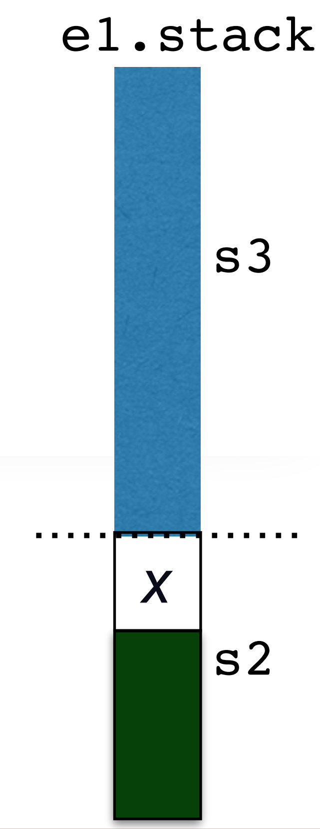
[http://jeanjacqueslevy.net/why3/graph/abs/scct/1-7/scc.html]

Assertions

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  assert {is_scc (elements s2)};
  assert {inter e.grays (elements s2) = empty by inter e.grays (elements s2) == empty};
  (max_int(), {blacks = add x e1.blacks; grays = e.grays;
     stack = s3; sccs = add (elements s2) e1.sccs;
     sn = e1.sn; num = set_max_int s2 e1.num}) end
```

Coq

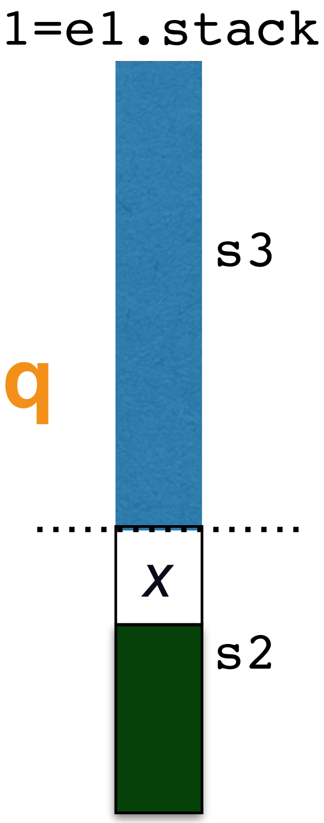
[http://jeanjacqueslevy.net/why3/graph/abs/scct/1-7/scc.html]

Assertions

```
assert {forall y. in_same_scc y x -> lmem y s2};
```

Coq

- proof by contradiction: $\exists y,\ \text{in\_same\_scc}\ y\ x\ \wedge\ y \notin s2$
- $\exists x'y',\ \text{reachable}\ x\ x'\ \wedge\ \text{edge}\ x'\ y'\ \wedge\ \text{reachable}\ y'\ y\ \wedge\ x' \in s2\ \wedge\ y' \notin s2$



Assertions

```
assert {forall y. in_same_scc y x -> lmem y s2};
```

Coq

- proof by contradiction: $\exists y, \ \text{in\_same\_scc}\ y\ x \ \wedge \ y \notin s2$
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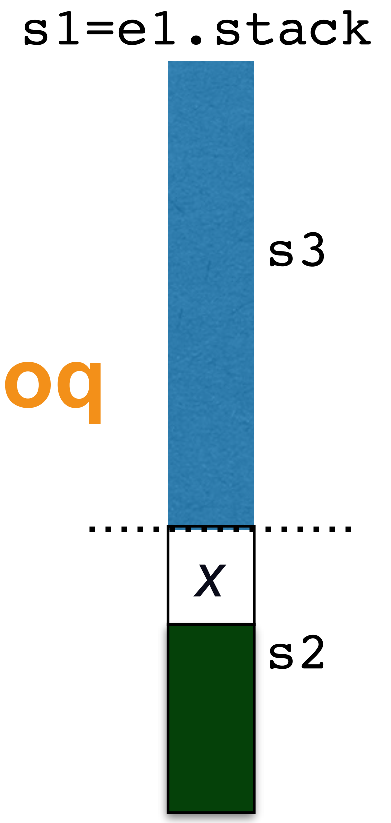
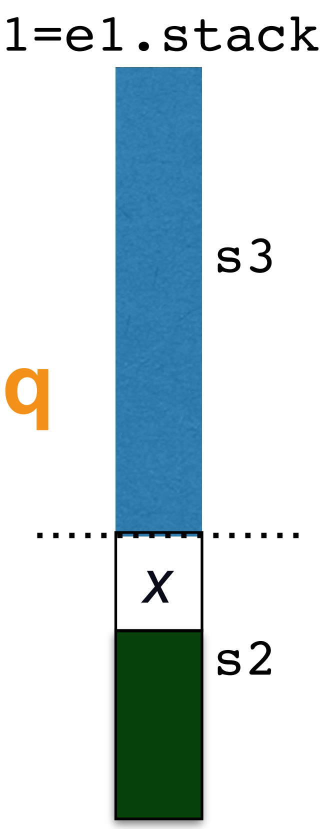
- 3 cases:



Assertions

```
assert {forall y. in_same_scc y x -> lmem y s2};
```

Coq



- proof by contradiction: $\exists y,\ \text{in\_same\_scc}\ y\ x \ \wedge\ y \notin s2$
- $\exists x' y',\ \text{reachable}\ x\ x' \ \wedge\ \text{edge}\ x'\ y' \ \wedge\ \text{reachable}\ y'\ y \ \wedge\ x' \in s2 \ \wedge\ y' \notin s2$
- 3 cases:

[1] y' is white

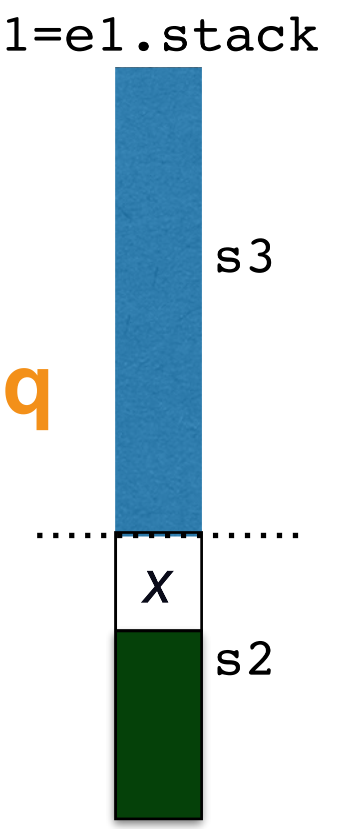
$x' = x$ then $y' \in \text{successors}\ x$ → y' is black

$x' \neq x$ then x' is black → $\neg\ \text{no\_black\_to\_white}\ b1\ g1$

Assertions

```
assert {forall y. in_same_scc y x -> lmem y s2};
```

Coq



- proof by contradiction: $\exists y,\ \ \text{in\_same\_scc}\ y\ x\ \wedge\ y \notin s2$
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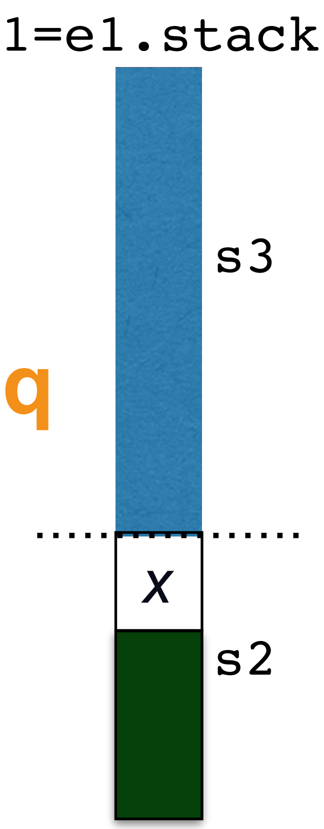
$x' = x$ then $y' \in \text{successors}\ x$ → y' is black

$x' \neq x$ then x' is black → $\neg\ \text{no\_black\_to\_white}\ \ b1\ g1$

[2] $y' \in \text{e1.sccs}$ then $\text{in\_same\_scc}\ \ y'\ x$ → x is black

Assertions




```
assert {forall y. in_same_scc y x -> lmem y s2};
```

← Coq

- proof by contradiction: $\exists y,\ \ \text{in\_same\_scc}\ y\ x\ \wedge\ y \notin s2$
- $\exists x' y',\ \ \text{reachable}\ x\ x'\ \wedge\ \text{edge}\ x'\ y'\ \wedge\ \text{reachable}\ y'\ y\ \wedge\ x' \in s2\ \wedge\ y' \notin s2$
- 3 cases:

[1] y' is white

$x' = x$ then $y' \in \text{successors}\ x$ → y' is black

$x' \neq x$ then x' is black → $\neg\ \text{no\_black\_to\_white}\ \ b1\ g1$

[2] $y' \in$ e1.sccs then $\text{in\_same\_scc}\ \ y'\ x$ → x is black

[3] $y' \in s3$ → $\text{rank}\ y'\ s1\ <\ \text{rank}\ x\ s1$ → $\text{e1.num}[y']\ <\ \text{e1.num}[x] = \text{e.num}[x] = n$

$x' = x$ then $y' \in \text{successors}\ x$ → $n1\ \leq\ \text{e1.num}[y']$

$x' \neq x$ then $\text{xedge\_to}\ \ s1\ (\text{Cons}\ \ x\ s3)\ \ y'$ ↗

Proof stats

provers	Alt-Ergo	CVC3	CVC4	Coq	E-prover	Spass	Yices	Z3	all	#VC	#PO
38 lemmas	2.35	0.23	5.79		0.66	0.75	0.21		9.99	77	38
split	0.09	0.2							0.29	6	6
add_stack_incr	0.01								0.01	1	1
add_blacks	0.01								0.01	1	1
set_max_int	0.02								0.02	1	1
dfs1	53.52	12.88	36.39	3.06	28.06			9.01	142.92	218	24
dfs	4.6	0.23	11.63					0.31	16.77	51	35
tarjan	0.44								0.44	16	6
total	61.04	13.54	53.81	3.06	28.72	0.75	0.21	9.32	170.45	371	112

[http://jeanjacqueslevy.net/why3/graph/abs/scct/1-7/scc.html]

Other systems

Coq / ssreflect

[cyril cohen, laurent théry, JJL]

- port in 1 week
- graphs and finite sets already in mathematical components
- problems with termination (hacky & higher-order)
- 920 lines

[http://github.com/CohenCyril/tarjan]

Isabelle / HOL

[stephan merz]

- port in 1 month
- use many strategies (metis, blast, sledgehammer)
- still problems with proving termination
- 31 pages

[http://jeanjacqueslevy.net/why3/graph/abs/scct/isa/Tarjan.pdf]

Fstar

`[kenji maillard, catalin hritcu]`

- start discuss with them
- Z3 single automatic prover
- ??

Conclusion

Future works

- library for formal proofs on graphs
- other graph algorithms
- **beyond** graphs …
- teaching formal methods on **test cases**
- **imperative** programs
- **Frama-C** embedded programs written in C
- readable formal proofs ?

[http://jeanjacqueslevy.net/why3]