Computing with Gödel's completeness theorem: Weak Fan Theorem, Markov's Principle and Double Negation Shift in action

Hugo Herbelin

CHoCoLa

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Computing with completeness proofs with respect to rich semantics

In LICS 1991, Ulrich Berger and Helmut Schwichtenberg initiate the idea of Normalisation-by-Evaluation.

In TLCA 1993, Ulrich Berger presents a paper on Program Extraction from Normalisation Proofs.

In CSL 1993, Catarina Coquand presents a formalised normalisation proof of simply-typed λ-calculus by transiting to Kripke semantics.

Then, a lot of a lot of works on constructive proofs of completeness wrt diverse "informative-enough" semantics: Kripke, Beth (Thierry Coquand, Jan Smith), point-free topology (Giovanni Sambin), phase semantics (Mitsuhiro Okada), Heyting algebras (Jim Lipton, Olivier Hermant), "glued" semantics (Thierry Coquand, Peter Dybjer), ... with applications to normalisation, with various connectives or features (Thorsten Altenkirch, Martin Hofmann, Philip Scott, Andreas Abel, Christian Sattler, ...). Works also on the analysis of reducibility/realisability/logical-relation proofs as "adequacy lemma $\circ$ escape lemma", as well as on type-directed partial evaluation (Olivier Danvy).

The case of Tarski semantics

Compared to "informative-enough" semantics such as Kripke, Beth, phase semantics (we can consider them as "effectful", in the sense of "carrying a state"), Tarski semantics is minimalistic. It simply replicates object syntax in the meta-language level:

$$
\begin{array}{lcl}
[\![P(t_1,\ldots,t_{a_f})]\!]^\sigma_{\mathcal{M}} & \triangleq & ([\![t_1]\!]^\sigma_{\mathcal{M}},\ldots,[\![t_{a_P}]\!]^\sigma_{\mathcal{M}}) \in \mathcal{M}_{\mathcal{P}}(P) \\
[\![\dot{\bot}]\!]^\sigma_{\mathcal{M}} & \triangleq & \bot \\
[\![A \dot{\to} B]\!]^\sigma_{\mathcal{M}} & \triangleq & [\![A]\!]^\sigma_{\mathcal{M}} \Rightarrow [\![B]\!]^\sigma_{\mathcal{M}} \\
[\![\dot{\forall} x\, A]\!]^\sigma_{\mathcal{M}} & \triangleq & \forall v \in \mathcal{M}_{\mathcal{D}}\, [\![A]\!]^{\sigma\cup[x\leftarrow v]}_{\mathcal{M}} \\
\\
\mathcal{T} \vDash A & \triangleq & \forall \mathcal{M}\, \forall \sigma\, ([\![\mathcal{T}]\!]^\sigma_{\mathcal{M}} \Rightarrow [\![A]\!]^\sigma_{\mathcal{M}})
\end{array}
$$

In spite of this sobriety (or weakness, depending on taste), Gödel, followed by many others (Henkin, Hasenjaeger, Beth, Hintikka, Kangers, Schütte , ...) could prove (in a classical metalanguage):

$$\mathcal{T} \vDash A \;\Rightarrow\; \mathcal{T} \vdash_{class} A$$

But while the proofs wrt "rich" semantics are mostly structural and are constructively mapping validity proofs into (normalised) object-syntax proofs reifying the validity proofs (thus supporting normalisation-by-evaluation), completeness proofs wrt Tarski semantics look more complicated...

First, a restriction for the rest of the talk

Completeness is commonly expressed under one of these forms:

- proof existence: $\mathcal{T} \models A \;\Rightarrow\; \mathcal{T} \vdash_{class} A$
- model existence: $\mathcal{T} \nvdash_{class} \dot{\bot} \;\Rightarrow\; \exists \mathcal{M} \exists \sigma\, [\![\mathcal{T}]\!]^{\sigma}_{\mathcal{M}}$
- proof or countermodel existence: $\mathcal{T} \vdash_{class} A \;\vee\; \exists \mathcal{M} \exists \sigma\, [\![\mathcal{T}]\!]^{\sigma}_{\mathcal{M}} \wedge [\![\dot{\neg} A]\!]^{\sigma}_{\mathcal{M}}$

Moreover, the model may be assumed:

- decidable: interpreting atoms as decidable propositions
- Boolean: interpreting atoms in $\mathbb{B}ool$
- Tarskian: interpreting atoms as arbitrary propositions

From the constructive point of view, each statement has its own specificities. We primarily focus for the talk on the first form, without requiring the model to be decidable or Boolean.

Specificities of completeness wrt Tarski semantics (proof existence form)

Apparently requires non-constructive features and/or choice/bar-induction axioms:

- requires Markov's principle (**MP**, i.e. $\neg\neg A \Rightarrow A$ for $A \in \Sigma_1^0$) according to Kurt Gödel (1957) and Georg Kreisel (1962),
- requires the law of excluded-Middle (**LEM**) according to Charles McCarty (2004) or Christian Espíndola (2016),
- requires the Ultrafilter Theorem according to Leon Henkin or more recently Christian Espíndola (2016),
- requires Weak König's Lemma (**WKL**) according to Stephen Simpson's textbook on the classical reverse mathematics of the subsystems of second order arithmetic,
- requires the Fan Theorem as suggested by Harvey Friedman (1975) and Wim Veldman (1976) and shown by Victor Krivtsov (2014),
- but actually requires no classical reasoning at all according to Jean-Louis Krivine (1996)!

Also apparently crucially rely on an enumeration of formulas (or to appeal to strong principles such as the ultrafilter theorem) and to not preserve the structure of the validity proof given as input. Why is it so? How crucial is it?

How to sort this out?

A first clarification by Stefano Berardi (1999) and Berardi-Valentini (2001): connectives have an impact on the logical strength

In the presence of falsity

If object-level $\dot{\bot}$ is interpreted as meta-level $\bot$, Markov's principle is required (formalised e.g. in Kirst-Forster-Wehr, 2020).

$\hookrightarrow$ This explains how Krivine bypasses the need for Markov's principle (no $\dot{\bot}$ in his language).

$\hookrightarrow$ The observation holds also for Beth and Kripke semantics and explains how Friedman (wrt Beth models) and Veldman (wrt Kripke models) bypass the need for Markov's principle (they use a different semantics for $\bot$).

In the presence of disjunction

If object-level $\dot{\vee}$ is interpreted as meta-level $\vee$, the proof cannot be constructive.

Wim Veldman's completeness proof wrt Kripke semantics (whose interpretation of $\dot{\vee}$ is Tarskian) suggest that Weak Fan Theorem is sufficient and Victor Krivtsov (2015) shows that it is needed. Moreover, the weakly classical part of the Weak Fan Theorem is identified by Josef Berger (2009) to be a principle called L_{fan}.

Can it be the piece of classical reasoning that disjunction requires?

Note: The above assumes either the presence of $\forall$, $\exists$, or of infinitely many atoms (otherwise, we are in propositional logic which is decidable).

More on the need for Markov's principle

Remind from linear logic the existence of two *falsity* connectives:

- 0 is the neutral element of the $\oplus$ (additive, positive) disjunction (i.e. the intuitive disjunction):

$$\frac{}{\Gamma, 0 \vdash \Delta}\ 0_L \qquad \text{no right introduction rule}$$

  its semantics is the *falsity* connective of the metalanguage

- $\bot$ is the neutral element of the $⅋$ (multiplicative, negative) disjunction:

$$\frac{}{\bot \vdash}\ \bot_L \qquad\qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \bot}\ \bot_R$$

  its semantics is an arbitrary formula of the metalanguage (leading to so-called "exploding" models)

In particular: it is the presence of 0 in the language (the positive form of *falsity*) which forces completeness to imply Markov's principle.

This suggests the slogan: it is the *positive* connectives which impact the logical strength.

(Incidentally, this also suggests that the same analysis should be done for a language highlighting the decomposition of connectives in linear logic, and within a meta-language that also makes explicit the decomposition of connectives given by linear logic.)

Second clarification: the need for excluded-middle

How can it be that McCarty and Espíndola derive **LEM** from completeness?

Answer: This is about the logical complexity of the theory under consideration.

If the theory $\mathcal{T}$ is allowed to be described by a formula of complexity S (typically Σ^0_1, but possibly another complexity), and *falsity* is interpreted like 0, then $\neg\neg A \Rightarrow A$ can be derived for any A of complexity S.

For instance, completeness for a recursively enumerable theory when *falsity* is interpreted in the standard (positive) way cannot prove more than Markov's principle (Kreisel).

To fix things, we now restrict ourselves to recursively enumerable theories.

Third clarification: Ultrafilter Theorem vs Weak Kőnig's Lemma vs Weak Fan Theorem

Stefano Berardi and Silvio Valentini (2001) again give an answer: exploiting Krivine's fully constructive proof, they were able to give a constructive proof of the Ultrafilter Theorem on countable domains.

Otherwise said: crossing with Stephen Simpson's result, Weak Kőnig's Lemma can (classically) be reinterpreted as the countable restriction of the Ultrafilter Theorem.

Digression on the logical structure of choice axioms

Exploiting the property that **WKL** lies at the intersection of the Ultrafilter Theorem and the axiom of Dependent Choices, Brede-Herbelin (2021) developed a generic view at choice and bar induction axioms connecting an "intensional" (or "operational", "finite", "actual", "effective") view and an "extensional" (or "observational", "ideal", "potential") view. They develop a "generalised dependent choice" scheme GDC_{ABT} over functions from A to B satisfying some filter T:

$$\underbrace{T \text{ coinductively } A\text{-}B\text{-approximable}}_{\textit{effective}} \implies \underbrace{T \text{ has an } A\text{-}B\text{-choice function}}_{\textit{observational}}$$

such that

$\mathsf{GDC}_{\mathbb{N}BT}$ is a formulation of Dependent Choices in B
$\mathsf{GDC}_{A\mathbb{B}ool\,T}$ is a formulation of the Ultrafilter Theorem on A
$\mathsf{GDC}_{\mathbb{N}\mathbb{B}ool\,T}$ is a formulation of Weak Kőnig's Lemma
GDC_{ABT} for "unary" filter T is a formulation to the full Axiom of Choice from A to B

and, a dual (asynchronous) "generalised bar induction" scheme GBI_{ABT} over functions from A to B satisfying some filter T:

$$\underbrace{T\ A\text{-}B\text{-barred}}_{\textit{observational}} \implies \underbrace{T\ A\text{-}B\text{-inductively barred}}_{\textit{effective}}$$

such that

$\mathsf{GBI}_{\mathbb{N}BT}$ is Bar Induction, which itself can be seen as a formulation of countable Zorn's Lemma over B
$\mathsf{GBI}_{\mathbb{N}\mathbb{B}ool\,T}$ is a formulation of Weak Fan Theorem

Towards a fourth clarification: WKL vs WFT and why incompatible equivalences with WFT

A glitch remains:

- Jean-Louis Krivine proved completeness wrt Tarski semantics fully constructively (reasoning in PA_2),
- refining Harvey Friedman and Wim Veldman, Victor Krivtsov showed that the Weak Fan Theorem is needed (reasoning in a variant of Weak Kleene-Vesley system), but following Josef Berger (2009), the Weak Fan Theorem is known to include a bit of classical logic.

How is this possible? It is that not everyone is talking about the same formulation of Weak Kőnig's Lemma and Weak Fan Theorem.

- WKL and WFT can be formulated in PA_2 in which case they are *intuitionistically* provable,[1]
- when paths are functions to $\mathbb{B}ool$, WKL and WFT include a bit of classical logic (respectively LLPO, i.e. de Morgan's law for Σ^0_1-formulas, and Berger's L_{fan}),
- this bit of classical logic remains needed in the presence of disjunction but otherwise not.

[1] for decidable trees in the WKL case

Propositional, Decidably Propositional, and Functional Weak Fan Theorem

There are **three distinct** possible definitions of a **set** of natural numbers in (second-order) constructive logic:

- a **subset**:

$$P : \omega \to \mathit{Prop}$$

- a **functional relation** mapping formulas to Booleans

$$R : \omega \times \mathbb{B}ool \to \mathit{Prop} \text{ such that } \forall n\, \exists! b\, R\,(n, b)$$

  or, equivalently, a **decidable subset** of formulas:

$$P : \omega \to \mathit{Prop} \text{ such that } \forall n\, (n \in P \vee n \notin P)$$

- a **function to** $\mathbb{B}ool$

$$f : \omega \to \mathbb{B}ool$$

This gives in turn **three different** kinds of **comprehension axioms**.

Contrastingly, in the presence of LEM, the first two formulations cannot be distinguished.

How the three representations of sets relate?

Obviously:

$$\omega \to \mathbb{B}ool$$

$$\Downarrow$$

$$R : \omega \times \mathbb{B}ool \to Prop \text{ such that } \forall n\, \exists! b\, R\,(n, b)$$

$$\Downarrow$$

$$\omega \to Prop$$

Map $f : \omega \to \mathbb{B}ool$ to $R(n, b) \triangleq (f(n) = b)$ which is trivially functional

Map $R : \omega \times \mathbb{B}ool \to Prop$ to $X(n) \triangleq R(n, \textsf{true})$

How the three representations of sets relate?

And also:

$$\omega \to \mathbb{B}ool$$

$$\mathsf{AC!}_{\mathbb{N},\mathbb{B}ool} \Uparrow$$

$$R : \omega \times \mathbb{B}ool \to Prop \text{ such that } \forall n\, \exists! b\, R\,(n, b)$$

$$\mathsf{LEM} \Uparrow$$

$$\omega \to Prop$$

Map $X : \omega \to Prop$ to $R(n, b) \triangleq (b = \mathsf{true} \Leftrightarrow X(n))$, this is functional by **LEM**

Map $\forall n\, \exists! b\, R\,(n, b)$ to a function by unique choice.

Propositional, Decidably Propositional, and Functional Weak Fan Theorem

Let T be an arbitrary predicate on $\mathbb{B}ool^*$ (finite sequences of Booleans)

$$\begin{array}{lcl} \mathsf{WFT}_{fun} & \triangleq & \forall f\, \exists n\, T(f_{|n}) \;\Rightarrow\; T^* \\[1em] \mathsf{WFT}_{fun-rel} & \triangleq & \forall R\, \text{functional}\, \exists l\, (l \approx R \wedge T(l)) \;\Rightarrow\; T^* \\[1em] \mathsf{WFT}_{pred} & \triangleq & \forall X\, \exists l\, (l \approx X \wedge T(l)) \;\Rightarrow\; T^* \end{array}$$

where $l \approx_n X$ (resp. $l \approx_n R$) expresses that l approximates the n first "values" of X (resp R):

$$\frac{}{\epsilon \approx X} \qquad \frac{l \approx X \quad X(|l|)}{l \cdot \text{true} \approx X} \qquad \frac{l \approx X \quad \neg X(|l|)}{l \cdot \text{false} \approx X}$$

$$\frac{}{\epsilon \approx R} \qquad \frac{l \approx R \quad R(|l|, b)}{l \cdot b \approx R} \qquad\qquad \begin{array}{lcl} f_{|0} & \triangleq & \epsilon \\ f_{|n+1} & \triangleq & f_{|n} \cdot f(n) \end{array}$$

and

$$T^* \;\triangleq\; \exists N\, \forall l\, (|l| = N \Rightarrow \exists l' \subset l\, T(l')) \qquad (= \text{uniformly barred})$$

Note: We do not care here about the logical complexity of T

Three forms of Weak Fan Theorem

Thus we have:

$$\mathsf{WFT}_{fun} \Rightarrow \mathsf{WFT}_{fun-rel} \Rightarrow \mathsf{WFT}_{pred}$$

While WFT_{fun} (considered in intuitionistic reverse mathematics) is told equivalent to the full Fan Theorem on finite (non-necessarily binary) "trees" (Iris Loeb 2005), $\mathsf{WFT}_{fun-rel}$ and WFT_{pred} are not equivalent to the corresponding formulation of the full Fan Theorem (based on Stephen Simpson's book).

WFT_{pred} is intuitionistically provable (in PA_2) and is enough to constructively prove completeness in the presence of $\Rightarrow$, $\wedge$, $\forall$ (over recursively enumerable theories) (Jean-Louis Krivine 1996) while disjunction requires a stronger version (Wim Veldman 1976, Victor Krivtsov 2014).

Combining Weak Fan Theorem and Markov's principle

We can add three variants of **WFT** capturing **MP** to the picture:

‘

$$\begin{tikzcd}[column sep=small]
 & & \mathsf{WFT}^{\neg\neg}_{fun} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
 & \mathsf{WFT}^{\neg\neg}_{fun-rel} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & \mathsf{WFT}_{fun} + \mathsf{MP} \arrow[dl, Rightarrow] \\
\mathsf{WFT}^{\neg\neg}_{pred} \arrow[dr, Rightarrow] & & \mathsf{WFT}_{fun-rel} + \mathsf{MP} \arrow[dl, Rightarrow] & \\
 & \mathsf{WFT}_{pred} + \mathsf{MP} & &
\end{tikzcd}$$

$$\begin{array}{lcl}
\mathsf{WFT}^{\neg\neg}_{fun} & \triangleq & \forall f\, \neg\neg\exists n\, T(f_{|n}) \;\Rightarrow\; T^* \\
\mathsf{WFT}^{\neg\neg}_{fun-rel} & \triangleq & \forall X \text{ decidable } \neg\neg\exists l\, (l \approx X \wedge T(l)) \;\Rightarrow\; T^* \\
\mathsf{WFT}^{\neg\neg}_{pred} & \triangleq & \forall X\, \neg\neg\exists l\, (l \approx X \wedge T(l)) \;\Rightarrow\; T^*
\end{array}$$

What do we need for disjunction?

Summary of former complementary results:

- Stefano Berardi and Silvio Valentini shows that disjunction cannot be treated purely intuitionistically.
- Wim Veldman (for Kripke semantics) and Victor Krivtsov (for Tarski semantics) have a proof using WFT_{fun}.
- WFT_{pred} is intuitionistically provable thus not sufficient.
- Josef Berger decomposes WFT_{fun} into a weak classical axiom L_{fan} and a pure choice axiom C_{fan}. Can L_{fan} be the missing piece?

What do we need for disjunction?
(recent results)

- Inspired by Andreas Abel and Christian Sattler normalisation-by-evaluation for Call-by-Push-Value (2019) and Ilik's intuitionistic continuation-based models (2013), Herbelin-Ilik (2022) handle disjunction by using DNS_T, where:

$$\begin{array}{lll} \text{(Double Negation Shift)} & \mathsf{DNS} & \triangleq \ \forall n\, \neg\neg A(n) \Rightarrow \neg\neg\forall n\, A(n) \\ \text{(Generalised DNS)} & \mathsf{DNS}_T & \triangleq \ \forall n\, ((A(n) \Rightarrow T) \Rightarrow T) \Rightarrow ((\forall n\, A(n)) \Rightarrow T) \Rightarrow T \quad (T \in \Sigma^0_1) \end{array}$$

  The purpose of Generalised DNS (Danko Ilik, 2011) is to provide DNS even in contexts where MP does not hold (otherwise $\mathsf{MP} + \mathsf{DNS}_T \Rightarrow \mathsf{DNS}$).

- Dominik Kirst (2022) shows in the context of Kripke completeness for an intuitionistic modal logic, where $\bot$ is interpreted as 0, that what is needed lies between a weak form of both WLEM and DNS, which he called WDNS, expressed equivalently as:

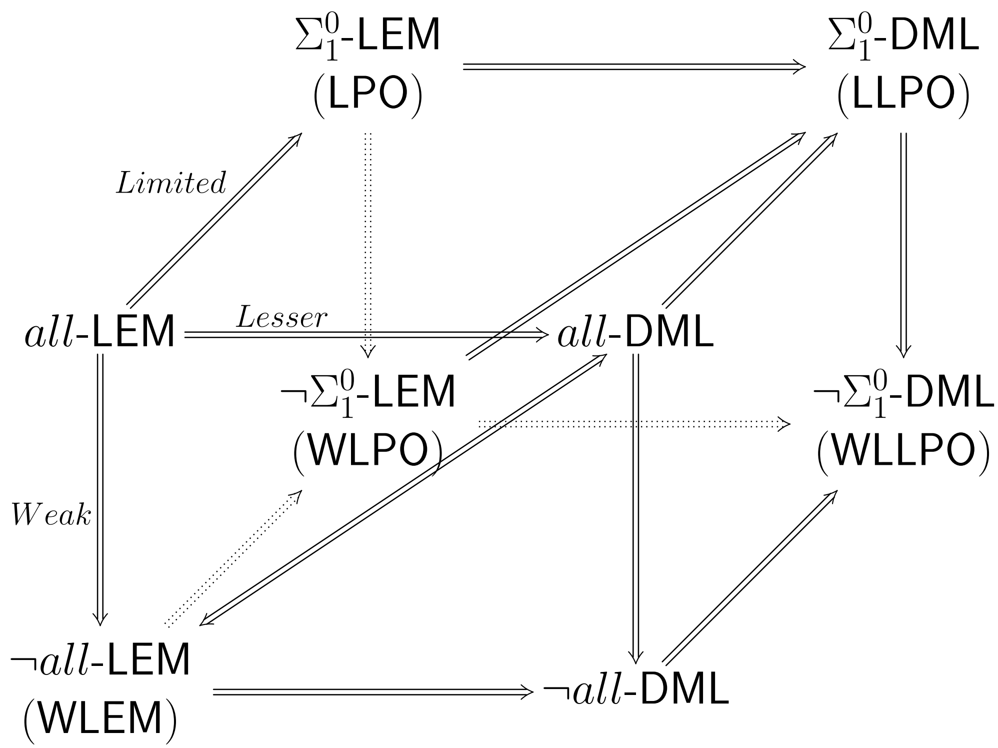
$$\begin{array}{lll} \text{(Weak Double Negation Shift)} & \mathsf{WDNS} & \triangleq \ \forall n\, \neg\neg(\neg A(n) \vee \neg B(n)) \Rightarrow \neg\neg\forall n\, (\neg A(n) \vee \neg B(n)) \\ & & \Leftrightarrow \ \neg\neg\forall n\, (\neg A(n) \vee \neg\neg A(n)) \end{array}$$

  and, either DNS again, or, the full Weak Excluded-Middle, expressed equivalently as:

$$\begin{array}{lll} \text{(De Morgan Law)} & \mathsf{DML} & \triangleq \ \neg\neg(\neg A \vee \neg B) \Rightarrow \neg A \vee \neg B \\ \text{(Weak Excluded-Middle)} & \mathsf{WLEM} & \triangleq \ \neg A \vee \neg\neg A \end{array}$$

- Just last week, Dominik Kirst actually showed that WDNS is enough for disjunction in both classical and modal propositional logic.

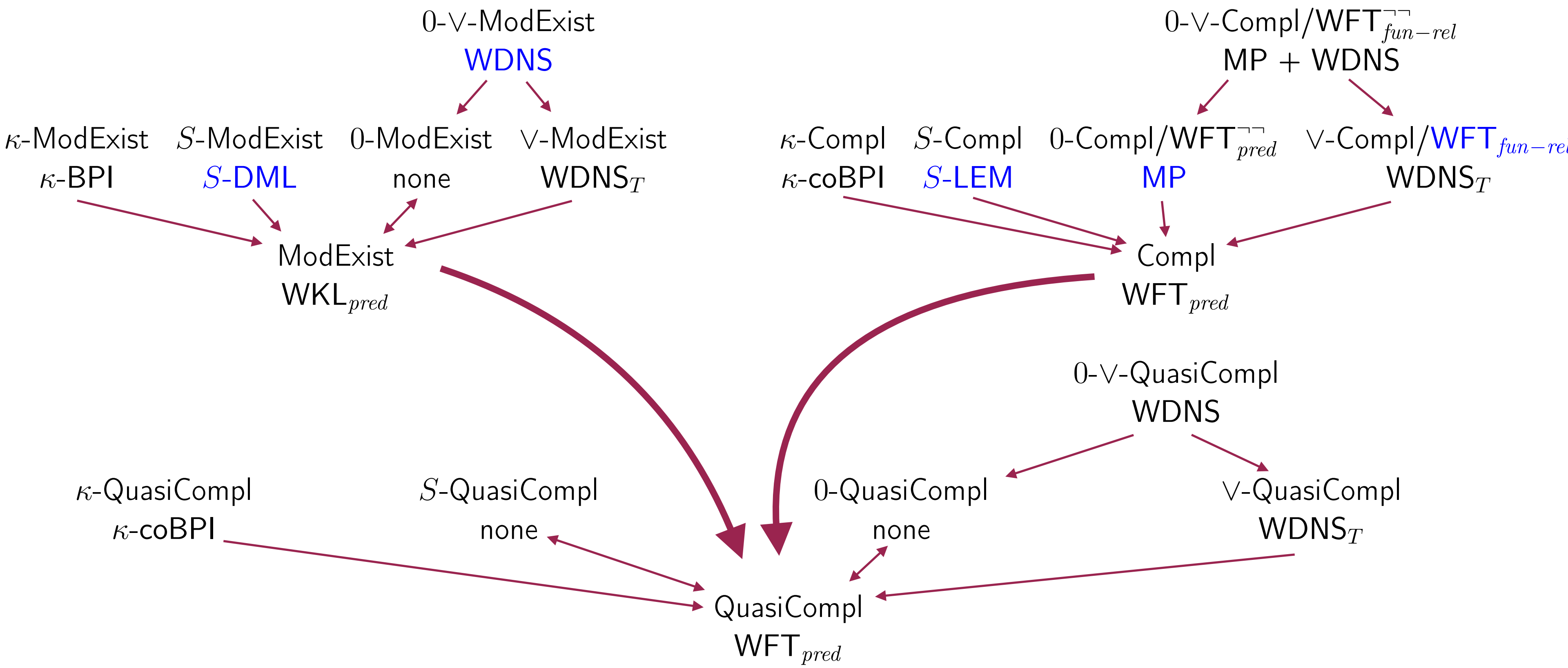
Positioning relatively to the basic cube of omniscience principles (e.g. Berger-Ishihara-Schuster 2012)



(Law of Excluded Middle)	S-LEM	$\triangleq$	$A \vee \neg A$	$(A \in S)$
(De Morgan's Law)	S-DML	$\triangleq$	$\neg(A \wedge B) \Rightarrow \neg A \vee \neg B$	$(A, B \in S)$

Thus, DNS and WDNS open in parallel a fourth dimension on top of all-LEM and WLEM/all-DML respectively.

Summary



ModExist	=	$\mathcal{T} \not\vdash_{class} \dot{\bot} \;\Rightarrow\; \exists \mathcal{M} \exists \sigma\; [\![\mathcal{T}]\!]^{\sigma}_{\mathcal{M}}$
Compl	=	$\mathcal{T} \vDash A \;\Rightarrow\; \mathcal{T} \vdash_{class} A$
QuasiCompl	=	$\mathcal{T} \vDash A \;\Rightarrow\; \neg\neg\mathcal{T} \vdash_{class} A$

0	=	with positive falsity
$\vee$	=	with positive falsity (or model decidability)
κ	=	theory of cardinal κ
S	=	theory of logical complexity S (e.g. Σ^0_1, or "all")

if not told explicitly: empty theory, $\Rightarrow$ and $\forall$

κ-BPI = Boolean Primer Ideal Theorem for cardinal κ

S-DML = $\neg\neg(\neg A \vee \neg B) \Rightarrow \neg A \vee \neg B$ (A, B of complexity S)

all-DML = WLEM; Σ^0_1-DML = LLPO; Σ^0_1-LEM = LPO

$\mathsf{WDNS}_T = \forall n\, \neg_T\neg_T(\neg_T A \vee \neg_T B) \Rightarrow \neg_T\neg_T \forall n\,(\neg_T A \vee \neg_T B)$ (T Σ^0_1)

$\mathsf{WDNS} = \mathsf{WDNS}_{\bot}$ (reminding that $\neg_T A$ is $A \Rightarrow T$)

blue = confirmed

otherwise, conjectured, possibly up to double negation

Remarks on the summary

- Most results in the literature do not distinguish between the predicative, functional relation and function versions of WKL, WFT, BPI.
- Most results in the literature do not consider separately 0 and $\vee$.
- κ-BPI can be compared to $\mathsf{GDC}_{\kappa\mathbb{B}ool}$: they differ in the same way as the negation of an inductive definition differ from the dual, coinductive definition (e.g. $\mathcal{T} \nvdash \dot{\bot}$ vs the coinductive definition of an inconsistency).
- Berger's L_{fan} (= "in an infinite tree with at most one infinite path, either the left or the right subtree is infinite") should morally be provably equivalent to Kirst's WDNS.

What do we need for existential quantification?

Surprisingly, $\exists$ does not require more than WFT_{pred} for Tarski completeness. Intuitively, this is because the models of existential quantification include the one-element models. Existential quantification alone does not ensure that a domain has more than one element. Disjunction is needed (e.g. $\forall x\,(x = 0 \vee \exists y\, x = y{+}1)$ and $\forall y\, 0 \neq Sy$ for $\mathbb{N}$). Or radically working with negative connectives.

Computing with completeness wrt Tarski semantics in the presence of positive falsity and disjunction

Importantly, MP and DNS_T (thus WDNS_T), as well as their conjunction, preserve the disjunction and existence properties (i.e. the constructive meaning of $\vee$ and $\exists$).

Also, both have known computational contents:

- unbounded search (Kleene) or exceptions (Herbelin 2010, justified by Thierry Coquand and Martin Hofmann's generalisation of Friedman's A-translation) for MP (see also Pierre-Marie Pédrot, 2019 for type theory),
- bar recursion (Clifford Spector 1976) or delimited control (Danko Ilik 2011) for DNS_T.

Recapitulation

- An analysis of the constructive content and intuitionistic reverse mathematics of completeness proofs.
- Following Stefano Berardi, an analysis of the weakly classical principles needed to handle positive connectives interpreted à la Tarski in completeness proofs:
  - positive falsity requires Markov Principle,
  - disjunction requires Weak (Generalized) Double Negation Shift.
- A clarification of an abundant and apparently contradictory literature on the topic.
- Keeping better in mind that Weak Kőnig's Lemma is the restriction of the Ultrafilter Theorem to countable domains.
- The observation of several variants of various strengths of the Weak Fan Theorem, some of them being purely intuitionistic.
- Suggesting that the same kind of analysis could be done in other contexts (e.g. for semantic realisability, as asked by Thomas Streicher in his notes on constructive logic and mathematics, 2000).

Computing with completeness wrt Tarski semantics

It seems that there are basically two kinds of completeness proofs wrt Tarski models:

- Henkin style: the data that a formula A belongs to the maximal extension of a consistent context computationally corresponds to giving a continuation reducing an inconsistency in the presence of A to an inconsistency without A.

  Eventually, the resulting derivation has roughly the form of a resolution proof (i.e. a tree of cuts inferring a contradiction from the axioms of the theory).

  See Herbelin-Ilik (draft, 2016, lastly revised 2022) for a detailed presentation highlighting the similarity with reify/reflect-based normalisation-by-evaluation.

  Two versions are given, one without DNS for $\dot\forall$, $\dot\rightarrow$, (negative) $\dot\bot$, $\dot\wedge$, $\dot\exists$ and one with DNS supporting also disjunction.

- Beth-Hintikka-Kanger-Schütte style, building an "universal" infinite proof of $\mathcal{T} \vdash A$ of which only the needed steps are eventually kept, according to what the proof of validity says, ending in a finite proof.

Another direction is to observe that Kripke semantics is obtained by forcing from Tarski semantics and to use a memory effect to simulate forcing in direct style: we have then the illusion to prove completeness with respect to Tarski semantics but, under the hood, the nice structural proof underlying completeness wrt Kripke semantics is used!

The constructive contents of Henkin's proof, main data structure

The computational role of the maximal theory construction is to add a formula to the theory under the condition that it preserves consistency (thus a continuation):

$$A \in \mathcal{S}_\omega \triangleq \exists n \, \exists \Gamma \, (\Gamma \subset \mathcal{S}_n \wedge \Gamma \vdash A)$$

where

$$\frac{}{\dot{\neg} A_0 \subset \mathcal{S}_0} \; I_0 \qquad \frac{\Gamma \subset \mathcal{S}_n}{\Gamma \subset \mathcal{S}_{n+1}} \; I_S \qquad \frac{\Gamma \subset \mathcal{S}_{2n}}{\Gamma, \; A(x_n) \dot{\to} \dot{\forall} x \, A(x) \; \subset \; \mathcal{S}_{2n+1}} \; I_{\forall}$$

$$\frac{\Gamma \subset \mathcal{S}_{2n+1} \qquad \mathcal{S}_{2n+1}, A \dot{\to} B \vdash \dot{\bot} \Rightarrow \mathcal{T}_0, \dot{\neg} A_0 \vdash \dot{\bot}}{\Gamma, \; A \dot{\to} B \; \subset \; \mathcal{S}_{2n+2}} \; I_{\Rightarrow}$$

where ϕ is an enumeration of formulas whose head connective is $\dot{\forall}$ or $\dot{\to}$, with $\phi(2n) = \dot{\forall} x \, A(x)$ in $I_{\forall}$ and $\phi(2n+1) = A \dot{\to} B$ in $I_{\Rightarrow}$.

The constructive contents of Henkin's proof, main lemma (with exploding nodes)

$$
\begin{array}{llll}
\downarrow_\sigma^A & : \ \mathcal{M}_0 \vDash_\sigma^e A & \Rightarrow & A[\sigma] \in \mathcal{S}_\omega \\
\downarrow_\sigma^{P(\vec{t})} & m & \triangleq & m \\
\downarrow_\sigma^{\dot\perp} & m & \triangleq & B\dot{O}T(m) \\
\downarrow_\sigma^{A \dot\to B} & m & \triangleq & AX\dot{I}MP_{A[\sigma]\dot\to B[\sigma]}(kont_\sigma^{A\dot\to B}(m)) \\
\downarrow_\sigma^{\dot\forall x\, A} & m & \triangleq & A\dot{P}P^{\dot\to}(AXDR\dot{I}NKER_{(\forall x\, A)[\sigma]}, \downarrow_{\sigma,x\leftarrow x_n}^A (m\, x_n)) \ where\ 2n = \ulcorner(\dot\forall x\, A)[\sigma]\urcorner
\end{array}
$$

$$
\begin{array}{llll}
\uparrow_\sigma^A & : \ A[\sigma] \in \mathcal{S}_\omega & \Rightarrow & \mathcal{M}_0 \vDash_\sigma^e A \\
\uparrow_\sigma^{P(\vec{t})} & q & \triangleq & q \\
\uparrow_\sigma^{\dot\perp} & q & \triangleq & \mathtt{flush}\, q \\
\uparrow_\sigma^{A \dot\to B} & q & \triangleq & m \mapsto \uparrow_\sigma^B (A\dot{P}P^{\dot\to}(q, \downarrow_\sigma^A m)) \\
\uparrow_\sigma^{\dot\forall x\, A} & q & \triangleq & t \mapsto \uparrow_{\sigma,x\leftarrow t}^A (A\dot{P}P^{\dot\forall}(q,t))
\end{array}
$$

where, for m proving $\mathcal{M}_0 \vDash_\sigma^e A\dot\to B$, the relative consistency proof $kont_\sigma^{A\dot\to B}(m)$ is defined by:

$$
\begin{array}{lll}
kont_\sigma^{A\dot\to B}(m) & : & (\mathcal{S}_{\ulcorner A[\sigma]\dot\to B[\sigma]\urcorner}, A[\sigma]\dot\to B[\sigma] \vdash \dot\perp) \Rightarrow (\mathcal{T}_0, \dot\neg A_0 \vdash \dot\perp) \\
kont_\sigma^{A\dot\to B}(m) & \triangleq & r \mapsto \mathtt{flush}\, (A\dot{P}P^{\dot\to}(PR\dot{O}J_2^{\dot\to}(r), \downarrow_\sigma^B (m\, (\uparrow_\sigma^A (PR\dot{O}J_1^{\dot\to}(r))))))
\end{array}
$$

and the reduction of an inconsistency from the maximal theory to the original theory is given by:

$$
\mathtt{flush}\ :\ \dot\perp \in \mathcal{S}_\omega \ \Rightarrow\ \mathcal{T}_0, \dot\neg A_0 \vdash \dot\perp
$$

The inference rules of the object logic

Primitive rules

$$\frac{|\Gamma'| = i}{\Gamma, A, \Gamma' \vdash A}\ \dot{\mathtt{ax}}_i \qquad \frac{\Gamma \vdash \dot{\neg}\dot{\neg}A}{\Gamma \vdash A}\ \dot{\mathtt{dn}}$$

$$\frac{\Gamma \vdash A \dot{\to} B \qquad \Gamma' \vdash A}{\Gamma \cup \Gamma' \vdash B}\ \dot{\mathtt{app}}^{\dot{\to}} \qquad \frac{\Gamma \vdash \dot{\forall} x\, A(x)}{\Gamma \vdash A(t)}\ \dot{\mathtt{app}}^{\dot{\forall}}_t$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \dot{\to} B}\ \dot{\mathtt{abs}}^{\Rightarrow} \qquad \frac{\Gamma \vdash A(y) \qquad y \text{ not in } \dot{\forall} x\, A(x), \Gamma}{\Gamma \vdash \dot{\forall} x\, A(x)}\ \dot{\mathtt{abs}}^{\forall}$$

Admissible rules

$$\frac{\Gamma, A(y) \dot{\to} \dot{\forall} x\, A(x) \vdash \dot{\bot} \qquad y \text{ not in } \dot{\forall} x\, A(x), \Gamma}{\Gamma \vdash \dot{\bot}}\ \dot{\mathtt{drinker}}_y$$

$$\frac{\Gamma, A \dot{\to} B \vdash \dot{\bot}}{\Gamma \vdash A}\ \pi_1^{\dot{\to}} \qquad \frac{\Gamma, A \dot{\to} B \vdash \dot{\bot}}{\Gamma \vdash \dot{\neg} B}\ \pi_2^{\dot{\to}} \qquad \frac{\Gamma \vdash \dot{\bot}}{\Gamma \vdash A}\ \dot{\mathtt{efq}} \qquad \frac{\Gamma \subset \Gamma' \qquad \Gamma \vdash A}{\Gamma' \vdash A}\ \dot{\mathtt{weak}}$$

and $\dot{APP}^{\dot{\forall}}$, $\dot{APP}^{\dot{\to}}$, $\dot{PROJ}_1^{\dot{\to}}$, $\dot{PROJ}_2^{\dot{\to}}$, $\dot{BOT}$, $\dot{AXDRINKER}_{\dot{\forall} x\, A}$, $\dot{AXIMP}_{A \dot{\to} B}$ lift rules to the maximal theory.

Example: the proof produced by the validity of $A_0 \triangleq X \dot{\to} Y \dot{\to} X$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\vdots\, p_0}{\dot{\neg} A_0, A_0 \vdash \dot{\bot}}
    }{\dot{\neg} A_0 \vdash \dot{\neg}(Y \dot{\to} X)}\,\pi_2^{\Rightarrow}
    \qquad
    \dfrac{}{\dot{\neg} A_0, Y \dot{\to} X \vdash Y \dot{\to} X}\,\dot{\mathtt{ax}}
  }{
    \dfrac{\dot{\neg} A_0, Y \dot{\to} X \vdash \dot{\bot}}{\dot{\neg} A_0 \vdash \dot{\neg} X}\,\pi_2^{\Rightarrow}
  }\,\dot{\mathtt{app}}^{\dot{\to}}
  \qquad
  \dfrac{
    \dfrac{\vdots\, p_0}{\dot{\neg} A_0, A_0 \vdash \dot{\bot}}
  }{\dot{\neg} A_0 \vdash X}\,\pi_1^{\Rightarrow}
}{
  \dfrac{
    \dfrac{\dot{\neg} A_0 \vdash \dot{\bot}}{\vdash \dot{\neg}\dot{\neg} A_0}\,\dot{\mathtt{abs}}\ \dot{\to}
  }{\vdash A_0}\,\dot{\mathtt{dn}}
}\,\dot{\mathtt{app}}^{\dot{\to}}
$$

where p_0 is:

$$
\dfrac{
  \dfrac{}{\dot{\neg} A_0 \vdash \dot{\neg} A_0}\,\dot{\mathtt{ax}}
  \qquad
  \dfrac{}{A_0 \vdash A_0}\,\dot{\mathtt{ax}}
}{\dot{\neg} A_0, A_0 \vdash \dot{\bot}}\,\dot{\mathtt{app}}^{\dot{\to}}
$$